Работа на конкурс Августа Мёбиуса

на тему

Generators and Relations of Symplectic Mapping Class Group of Rational 4-manifolds

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Москва
2017

Contents

1 Abstract

In this paper I describe a natural presentation of the symplectic mapping class group $SMap(X,\omega) := \pi_0(Symp(X,\omega))$ of rational 4-manifolds X in the case when the symplectic form ω has type D. Explicitly, I describe the conditions on the cohomology class $[\omega]$ which characterize the type D and show that in this case there exists a set of generators of the group $SMap(X,\omega)$ which are symplectic Dehn twists along ω-Lagrangian spheres S_i with the incidence graph D_l. In particular, this implies that the group $SMap(X,\omega)$ is the quotient of the braid group $\mathsf{Br}(\mathrm{D}_l)$ of type D. Finally, I find the set of additional relations, which gives the desired presentation of the group $SMap(X,\omega)$.

2 Introduction

This paper is devoted to description of the symplectic mapping class groups $\pi_0(Symp(X, \omega))$ (SyMCG) of rational 4-manifolds (multiple blow-ups of $\mathbb{CP}^2$). This work is a continuation of my diploma thesis [4]. Recall that a 4-manifold is called **rational** if it is diffeomorphic to $S^2 \times S^2$ or to a (multiple) blow-up of $\mathbb{CP}^2$. Since in the case $X = S^2 \times S^2$ the SyMCG is well-understood, we ignore this case and denote by X_l the l-fold blow-up of $\mathbb{CP}^2$.

It appears that, unlike the case of mapping class groups of surfaces, the SyMCG depends substantially on the symplectic form ω. On the other hand, it is known that the symplectic mapping class group of a rational 4-manifold X depends only on the cohomology class $[\omega]$ of the form.

Also, it is known (see [26]) that if the number of blow-up is $l < 5$, then a symplectomorphism homotopic to identity is symplectically isotopic to identity, and the SyMCG is finite. Therefore, it is interesting to find the cases when this group is rather big, and give a description of the SyMCG (for example, in a form of a nice presentation) in those cases.

In my diploma thesis [4] I described the dependence of SyMCG on the cohomology class $[\omega]$. In particular, I have shown that for generic symplectic form ω the SyMCG is trivial, and found two important cases, denoted by E_l $(l = 5, \ldots, 8)$ and D_{l-1} $(l = 5, \ldots, 9)$, in which one can expected a big SyMCG. In particular, I have shown that the image of the symplectic MCG in the smooth MCG is a reflection group (= Weyl group) of type E_l or resp. D_{l-1}, and that the SyMCG $\pi_0(Symp(X, \omega))$ admits a surjective homomorphism from the braid group of the same type E_l or resp. D_{l-1}.

The goal of this paper is to give a complete description of the SyMCG in the case D.

The first result is the complete description of the type D. It appears that in the case of large number of blow-ups ($l \geq 10$) the SyMCG is generated not only by "usual" symplectic Dehn twists along Lagrangian spheres $S \subset (X, \omega)$, but also so called "elliptic twists", see [48] for details. The construction of such an elliptic twist involves certain embedded (-1)-tori C such that $\xi \cdot [C] \leq 0$ and $c_1(X) \cdot [C] = -1$. In **Subsection 3.1** I describe the condition when no such tori exist.

Next, I reduce the calculation of the SyMCG to the calculation of the fundamental group of certain special divisor in moduli space blow-ups of $\mathbb{CP}^2$, thus it makes possible to find a natural geometric presentation of the group $\pi_0(Symp(X, \omega))$. In our cases symplectic mapping class group is a quotient of the braid group $Br(\mathrm{D}_l)$ and generators are symplectic Dehn twists along Lagrangian spheres.

For the case D_l I also describe relations on generators. Hence, we find a desirable presentation.

3 Preliminary results

Let us first recall the definitions which we shall use in this paper. To describe these problems in more details, let us introduce some notation. Let X be a compact manifold. Then

- $Z^2(X)$ denotes the space of closed 2-forms on X;
- $\Omega(X)$ to denotes the space of symplectic forms ω on X;
- $\Omega(X,\xi)$ denotes the space of symplectic forms ω on X in a cohomology class $\xi \in H^2(X,\mathbb{R})$;
- $\mathit{Diff}(X)$ denotes the group of diffeomorphisms of X,
- $\mathit{Diff}_0(X)$ denotes the connected component of the identity of diffeomorphism group $\mathit{Diff}(X)$; thus $\mathit{Diff}_0(X)$ is the isotopy group of X;,
- $\Gamma(X) := \mathit{Diff}(X)\big/\mathit{Diff}_0(X) = \pi_0(\mathit{Diff}(X))$ denotes **diffeotopy group**, another name of $\Gamma(X)$ is **mapping class group** of a smooth compact manifold X,
- $\mathit{Diff}(X,\xi)$ denotes the subgroup of diffeomorphism group $\mathit{Diff}(X)$ which consists all diffeomorphisms preserving a given cohomology class $\xi \in H^2(X,\mathbb{R})$, thus $\mathit{Diff}(X,\xi)$ is the stabilizer of $\xi \in H^2(X,\mathbb{R})$ in $\mathit{Diff}(X)$;
- $\mathit{Symp}(X,\omega)$ denotes the symplectomorphism group of a symplectic manifold with fixed symplectic form (X,ω), thus $\mathit{Symp}(X,\omega)$ is the stabiliser of an element ω w.r.t. the action of $\mathit{Diff}(X)$ on the space $\Omega(X)$;
- $\mathit{Symp}_0(X,\omega)$ denotes the connected component of the identity of symplectomorphism group $\mathit{Symp}(X)$, which is called the symplectic isotopy group of X;
- $\mathit{SMap}(X,\omega) := \mathit{Symp}(X,\omega)\big/\mathit{Symp}_0(X,\omega) = \pi_0(\mathit{Symp}(X,\omega))$ denotes the **symplectic mapping class group** of a compact symplectic manifold (X,ω), also it is called **symplectotopy group**,

Now let us define some terms concerning almost complex structures and pseudoholomorphic curves.

Definition 3.1. • An endomorphism $J : TX \to TX$ of the tangent bundle such that $J^2 = -Id_{TX}$ is called an **almost complex structure** on a manifold X.

- If $\omega(v, Jv) > 0$ for every non zero tangent vector v we will say that symplectic form ω on X **tames** an almost complex structure J .
- If for an almost complex structure J on X the following is true: in a neighborhood of every point $x \in X$ there exists a local *complex* coordinate system $(z_1,\dots,z_n)$ such that $J\frac{\partial}{\partial z_j} = \mathbf{i}\frac{\partial}{\partial z_j}$; then we will say that J is an integrable.
- An integrable almost complex structure is called a **complex structure**.

Definition 3.2. S is a closed surface and J an almost complex structure on X, then

- a C^1-smooth map $u : S \to X$ is called **J-holomorphic** (or, if J is not specified, **pseudoholomorphic**) if the following true: there exists a complex structure J_S on S such that $du : TS \to TX$ is (J_S, J)-linear, i.e.,

$$du \circ J_S = J \circ du : T_zS \to T_{u(z)}X \quad \text{for every} \quad z \in S;$$

- **pseudoholomorphic** or **J-holomorphic** curve is an image $u(S)$ of a non-constant pseudoholomorphic map.

Also explain the following notation that we use in our work. For this aim fix some sufficiently large $k \in \mathbb{N}$ and some α with $0 < \alpha < 1$.

- $\mathcal{J}(X)$ denotes the space of all almost $C^{k,\alpha}$-smooth complex structures tamed by *some* symplectic form (let us notice that there is no fixed symplectic form, every complex structure can be tamed by its own symplectic form)
- $\mathcal{J}(X,\xi)$ denotes the space of that $J \in \mathcal{J}(X)$ which are tamed by symplectic form ω from $\Omega(X,\xi)$
- $\Omega\mathcal{J}(X)$ denotes the space of pairs (ω, J): ω is a symplectic form that tames J an almost complex structure J;
- $\Omega\mathcal{J}(X,\xi)$ denotes the set of those pairs $(\omega, J) \in \Omega\mathcal{J}(X)$ for which $\omega \in \Omega(X,\xi)$;
- $\mathcal{J}^{int}(X) \subset \mathcal{J}(X)$ denotes the subspace of integrable almost complex structures;
- $\mathcal{J}^{int}(X,\xi) \subset \mathcal{J}(X,\xi)$ denotes the subspace of that almost complex structures J which are tamed by some ω from $\Omega(X,\xi)$;
- the diffeomorphism group $Diff(X)$ naturally acts on all spaces above;

Let us recall that the cohomology space $H^2(X,\mathbb{R})$ has the Lorentzian signature type. Our standard basis of the cohomology space $H^2(X,\mathbb{R})$ is $L, E_1, \dots, E_l$, such that L is the homology class of line in $\mathbb{CP}^2$ with $L^2 = 1$ and E_i is homology class of the i-th exceptional curve with $E_i^2 = -1$. Consider the cohomology classes $\xi \in H^2(X,\mathbb{R})$ such that $\xi^2 > 0$, these classes forms the double cone, fix one cone half and call it a positive cone $\mathcal{K}_+$. Then $Diff_+$ is the group of deffeomorphisms preserving the orientation of X and preserving positive cone. Notice that in case of rational 4-dimensional manifolds there only two cases for that it is possible to change orientation: $S_2 \times S_2$ (quadric) and $\mathbb{CP}2$ blown up at one point.

Now denote by $\Gamma_+(X) \subset \Gamma(X)$ the mapping class group of X preserving orientation and positive cone $\mathcal{K}_+(X)$. Consider the natural homomorphism $\rho : \Gamma_+(X) \to Aut(H^2(X,\mathbb{Z}))$, that is induced by action of the group of diffeomorphism that preserves $Diff_+$ on cohomology group $H^2(X,\mathbb{Z})$, and the following exact sequence

$$1 \to \Gamma_{\blacksquare}(X) \to \Gamma_+(X) \to \Gamma_W(X) \to 1,$$

where $\Gamma_{\blacksquare}(X)$ is the kernel of the map $\rho : \Gamma_+(X) \to Aut(H^2(X,\mathbb{Z}))$ and Γ_W is the image of the map $\rho : \Gamma_+(X) \to Aut(H^2(X,\mathbb{Z}))$

Let us formulate the following theorem about the sequence above.

Theorem 3.1. [47] *The exact sequence*

$$1 \to \Gamma_{\blacksquare}(X) \to \Gamma_+(X) \to \Gamma_W(X) \to 1,$$

splits and $\Gamma_+(X)$ *is isomorphic to semi-direct product* $\Gamma_{\blacksquare}(X) \rtimes \Gamma_W(X)$

Let we have rational 4-dimensional manifold (X,ω') with symplectic form ω' on it, ω' lies in cohomology class $\xi' \in H^2(X,\mathbb{R})$. Recall that there exists a diffeomorphism F, such that F maps (X,ω') into (X,ω) and ω lies in cohomology class $\xi \in H^2(X,\mathbb{R})$ and for this cohomology class the following inequalities are true:

$$\xi \cdot S'_{123} > 0, \quad \xi \cdot S_{i,i+1} > 0,\ i = 1, \dots, l-1, \quad \xi \cdot E_l > 0, \tag{3.1}$$

where $S'_{123} = L - (E_1 + E_2 + E_3)$ for $i = 0$, $S_{i,i+1} = E_i - E_{i+1}$ for $i = 1, \dots, l-1$ and E_l in case of s_l.

Remark 3.1. This diffeomorphism F corresponds to the changing of blowing down order.

Moreover, the image of the mapping class group preserving positive cone $\Gamma_+(X)$ into the group of automorphisms of 2-cohomology space $Aut(H^2(X,\mathbb{Z}))$ is the Coxeter-Weyl group $\mathsf{W}(\mathcal{S}(X))$ and the Coxeter system $\mathcal{S}(X)$ has the type $S(X)) = BE_{l+1}$ and the following Coxeter graph

Definition 3.3. Let ω be a symplectic form, then **symplectic cone** $\mathcal{K}_\omega$ is the subset of all cohomology classes $\xi \in H^2(X,\mathbb{R})$, which are represented by any ω.

Corollary 3.2. *Consider a symplectic form ω and a Weyl chamber $\mathcal{C} \subset \mathcal{K}_\omega$ of the group $\Gamma_W(X)$. Then there exists a diffeomorphism F such that $F_*[\omega]$ lies in the closure $\bar{\mathcal{C}}$.*

Corollary 3.3. *Consider a rational 4-dimensional manifold (X,ω). Then its group of symplectomorphisms $Symp(X,\omega)$ depends only on cohomology class $[\omega]$.*

For every a rational symplectic 4-manifold with the symplectic form ω in the cohomology class ξ there is the following principal bundle:

$$1 \to Symp(X,\omega) \to Diff(X,\xi) \to \Omega(X,\xi) \to 1$$

Because for an action of the group $Diff(X,\xi)$ the space $\Omega(X,\xi)$ is an orbit and the group of symplectomorphisms $Symp(X,\omega)$ is the stabilizer of the form ω. Also, we have the associated (with short sequence above) long exact sequence:

$$\begin{aligned} \ldots \longrightarrow \pi_1(Symp(X,\omega)) \longrightarrow \pi_1(Diff(X,\xi)) \longrightarrow \pi_1(\Omega(X,\xi)) \xrightarrow{\partial} \\ \xrightarrow{\partial} \pi_0(Symp(X,\omega)) \longrightarrow \pi_0(Diff(X,\xi)) \longrightarrow \pi_0(\Omega(X,\xi)). \end{aligned} \tag{3.2}$$

Using the long exact sequence (3.2) it is easy to prove [4] the next lemma:

Lemma 3.4. *The image of the symplectic mapping class group $\pi_0(Symp(X,\omega))$ in the mapping class group $\pi_0(Diff(X,\xi))$ is $\Gamma_W(X,\xi)$, where $\Gamma_W(X,\xi)$ denotes the stabilizer of the element ξ in the group $\Gamma_W(X)$.*

Now let us formulate the theorem describing the group $\Gamma_W(X,\xi)$.

Theorem 3.5. [4] **(i)** *The group $\Gamma_W(X,\xi)$ is a finite reflection group $\mathsf{W}(\mathcal{S}(X,\xi))$ corresponding to the following Coxeter system $\mathcal{S}(X,\xi)$: $s_i \in \mathcal{S}(X,\xi)$ iff $s_i \in \mathcal{S}(X)$ and s_i preserve ξ.*

Remark 3.2. As we know that $\Gamma_W(X,\xi) = \mathsf{W}(\mathcal{S}(X,\xi))$ is a finite reflection group, then $\mathcal{S}(X,\xi))$ is

(ii) *The Coxeter system $\mathcal{S}(X,\xi)$ is a subset of Coxeter system $\mathcal{S}(X)$ and consists exactly of those cohomology classes, which are orthogonal to the cohomology class ξ.*

Lemma 3.6. *The set of roots of the Coxeter system $\mathcal{S}(X,\xi)$ coincides with the set of spherical (-2)-classes $S \in \mathcal{E}_{(-2)}$ orthogonal to ξ, these spherical (-2)-classes $S \in \mathcal{E}_{(-2)}$ realized as (-2)-Lagrangian spheres in X*

Definition 3.4. Consider the quotient space $\mathcal{J}^{int}(X,\xi)/\mathit{Diff}(X,\xi)$. This space is **the moduli space of polarised rational surfaces**. We denote it as $\mathfrak{M}(X,\xi)$.

Let us notice that for different points $J \in \mathcal{J}^{int}(X,\xi)$ their stabilizer can be different. Namely, the stabilizer group is the automorphism group $Aut(X,J)$ and there are obvious examples when this group is non-trivial. On the other hand, for $l \geq 4$ and for generic J the automorphism group is trivial. This means that that the quotient $\mathcal{J}^{int}(X,\xi)/\mathit{Diff}(X,\xi)$ is not a principle bundle, and the long exact sequence of homotopy group could fail to exist.

Now consider the kernel $\mathit{Diff}_{\blacksquare}$ of the homomorphism $\mathit{Diff}_{+} \to \Gamma_W(X)$.

Definition 3.5. The moduli space of polarised marked rational complex surfaces $\widetilde{\mathfrak{M}}(X,\xi)$ is a quotient space $\mathcal{J}^{int}(X,\xi)/\mathit{Diff}_{\blacksquare}(X,\xi)$.

Theorem 3.7. *In the diagram*

$$\begin{array}{ccccccccc}
\pi_1(\mathit{Diff}(X,\xi)) & \xrightarrow{\phi_1} & \pi_1(\Omega(X,\xi)) & \xrightarrow[\partial]{\phi_2} & \pi_0(\mathit{Symp}(X,\omega)) & \xrightarrow{\phi_3} & \pi_0(\mathit{Diff}(X,\xi)) & \xrightarrow{\phi_4} & \pi_0(\Omega(X,\xi)) \\
\uparrow = & & \phi_6 \uparrow \cong & & \uparrow & & \uparrow & & \\
\pi_1(\mathit{Diff}(X,\xi)) & \xrightarrow{\phi_5} & \pi_1(\mathcal{J}(X,\xi)) & & \phi \, \vdots & & \| \, = & & \\
\uparrow = & & \phi_8 \uparrow & & \vdots & & \| & & \\
\pi_1(\mathit{Diff}(X,\xi)) & \xrightarrow{\phi_7} & \pi_1(\mathcal{J}^{int}(X,\xi)) & \xrightarrow{\phi_9} & \pi_1(\mathfrak{M}(X,\xi)) & \xrightarrow[\partial]{\phi_{10}} & \pi_0(\mathit{Diff}(X,\xi)) & &
\end{array} \tag{3.3}$$

the homomorphism ϕ is an isomorphism.

Lemma 3.8. *If $A \in H_2(X,\mathbb{Z})$ and $A^2 < 0$ then there exists at most one J-holomorphic curve C, which realizes the class A.*

Theorem 3.9. [4] *Let ξ can be realized as cohomology class of some symplectic form and ξ satisfies the period conditions 3.1. Then the group $\pi_0\big(\mathit{Symp}(X,\xi)\big)$ is trivial.*

Lemma 3.10. *For every structure $J \in \mathcal{J}(X)\backslash\mathcal{J}(X,\xi)$ there exists an irreducible J-holomorphic curve C such that $[C]^2 < 0$ and $\int_C \xi \leq 0$.*

By this lemma in the case $A^2 < 0$ the moduli space $\mathcal{M}(X,A)$ of pseudoholomorphic curves of the homology class A is identified with the space of all those structures $J \in \mathcal{J}(X)$ such that there exists an irreducible J-holomorphic curve C with the class $[C] = A$.

Theorem 3.11. *Assume that $A^2 < 0$.*

(i) *The space $\mathcal{M}(X,A)$ is a Banach submanifold of $\mathcal{J}(X)$, and its real codimension equals to $-(c_1(X)\cdot A + A^2) \in \mathbb{Z}_+$.*

(ii) *The expected ("arithmetic") genus of a curve C in the class A is $g_{ar}(A) = \frac{A^2 - c_1(X)\cdot A}{2} + 1$.*

(iii) *The space $\mathcal{M}(X,A)$ has real codimension 2 in $\mathcal{J}(X)$ the following cases:*

(1) curve C in the class A is a rational (-2)-curve and $\int_C \xi \leq 0$;

(2) C is an elliptic (-1) curve and $\int_C \xi \leq 0$.

Lemma 3.12. *Let X be diffeomorphic to $\mathbb{CP}^2$ blown up in $l \leq 9$ points, and J a tamed almost complex structure on X. Then there exists no J-holomorphic curve C on X with $C^2 = -1$ and $c_1(X)\cdot C = -1$.*

3.1 Homological conditions on $[\omega]$ for absence of toric twists

We are interested in studying $\pi_1(\mathcal{J}(X,\xi))$. By the **Theorem 3.10** the space $\mathcal{J}(X,\xi) \subset \mathcal{J}(X)$ is the complement to the set of the structures J for that there is an irreducible J-holomorphic curve C such that $[C]^2 < 0$ and $\int_C \xi \leq 0$. Also, we know that $\mathcal{J}(X)$ is simply connected. It follows the group $\pi_1(\mathcal{J}(X,\xi))$ is generated by paths around the component of $\mathcal{J}(X)\backslash\mathcal{J}(X,\xi)$, which have codimension 2. Now by **Theorem 3.11** those components correspond to either (-2)-spheres or to (-1)-tori.

The paper [48] describes a possible contribution to the SyMCG by (-1)-tori. It was shown the following:

- Assume that C is an ω-symplectic (-1) torus in X. Then there exists a deformation ω' of the symplectic structure such that $\int_C \omega'$ is non-positive.
- Set $\xi := [\omega]$ and $\xi' := [\omega']$. Then the complement $\mathcal{J}(X,\xi)\backslash\mathcal{J}(X,\xi')$ contains the space $\mathcal{J}(X;[C])$ of those structures $J \in \mathcal{J}(X)$ for which there exists a non-singular irreducible J-holomorphic curve in the class $[C]$. (The complement $\mathcal{J}(X,\xi)\backslash\mathcal{J}(X,\xi')$ could contain other strata.)
- We denote by $\mathcal{J}(X,\xi,[C])$ the intersection $\mathcal{J}(X,\xi) \cap \mathcal{J}(X,[C])$. Then $\mathcal{J}(X,\xi,[C])$ is a non-empty Banach submanifold in $\mathcal{J}(X,\xi)$ of real codimension 2. In particular, we obtain an element in $\pi_1\mathcal{J}(X,\xi')$ which is represented by a small loop in $\mathcal{J}(X,\xi)$ around $\mathcal{J}(X,[C])$.
- This element defines a non-trivial element in the symplectic mapping class group $\pi_0 Symp(X,\omega')$.

Moreover, if X_l is $\mathbb{CP}^2$ blown up at $l \leq 10$ points, then there are no elements in $\pi_0 Symp(X_l,\omega)$ which arise from (-1)-tori.

At this point we give a formal definition of the cases for which we can compute (or at least understand) the symplectic mapping class group.

Definition 3.6. Let X be a rational 4-manifold, $L; E_1,\ldots,E_l$ the standard basis of X, ω a symplectic form on X, and $\xi := [\omega]$ its cohomology class. Assume additionally that $\xi = [\omega]$ satisfies the inequalities (3.1), possibly non-strict.

- We say that the symplectic form ω and cohomology class ξ have type $\mathbf{D}_l$, when in the period conditions one has the following strict (in)equalities:

$$\xi \cdot S'_{123} = 0, \quad \xi \cdot S_{12} > 0, \quad \xi \cdot S_{i,i+1} = 0,\ i = 2,\ldots,l-1; \tag{3.4}$$

  and there are *no ω-symplectic (-1)-tori.*

- We say that the symplectic form ω and cohomology class ξ has type $\mathbf{E}_l$, when in period conditions one has only equalities:

$$\xi \cdot S'_{123} = 0, \quad \xi \cdot S_{i,i+1} = 0,\ i = 1\ldots,l-1 \tag{3.5}$$

Remark 3.3. Notice that the condition $\xi \cdot E_l > 0$ is always satisfied since E_l is a (-1)-sphere, whereas all other classes $S_{\ldots}$ are (-2)-spheres.

Theorem 3.13. *Let X be $\mathbb{CP}^2$ blown up in l points, and C a symplectically embedded torus such that $C^2 = -1$ and $c_1(X) \cdot C = -1$. Then $l \geq 10$ and there exists a geometric basis $L; E_1, \ldots, E_l$ in which the holomogy class of C is*

$$[C] = 3mL - \textstyle\sum_{i=1}^{9} mE_i - E_{10}. \tag{3.6}$$

Remark 3.4. A surface C is symplectically embedded in X if there exists a symplectic form ω on X such that C is ω-symplectic. In this case we take the orientation on C defined by ω. We also assume that the geometric basis $L; E_1, \ldots, E_l$ is represented by ω-symplectic spheres.

Proof. Fix a geometric basis $L, E_1, \ldots, E_l$. Then we can write $[C] = aL - \sum_i b_i E_i$ with coefficients $a := C \cdot L$ and $b_i := C \cdot E_i$. Find an ω-tamed almost complex structure J such that C is J-holomorphic. Notice that we can perturb J outside C and make it generic enough. Consequently, every class L, E_i can be realized by a J-holomorphic rational curve, still denoted by L or E_i. In this situation Gromov's theory ensures that a and every b_i are non-negative integers.

The property above holds for coefficients a, b_i of the class $[C]$ in any geometric basis. The idea of the proof is to find such a geometric basis $L', E'_1, \ldots, E'_l$ in which the coefficient a' is minimal possible. Let us describe the procedure we make use of. First, we notice that we can arbitrarily permute classes $E_1, \ldots, E_l$. So we shall always assume that the coefficients b_i form a decreasing sequence: $b_1 \geq b_2 \geq \cdots$.

Next, consider the reflection s_0 across the wall orthogonal to the class $S'_{123} = L - E_1 - E_2 - E_3$. Set $\delta = -C \cdot S'_{123} = b_1 + b_2 + b_3 - a$. Recall that s_0 is realized by a diffeomorphism and sends every geometric basis into another geometric basis. The change of the coefficients of $[C]$ under s_0 is as follows: $a \mapsto a' = a - \delta$, $b_i \mapsto b'_i = b_i - \delta$ for $i = 1, 2, 3$, the coefficients b_i with $i \geq 4$ remain unchanged. So if $\delta > 0$ we can decrease the coefficient a. Since a is integer and non-negative, after repeating this procedure we come to the case when further decreasing of a is impossible. This situation is characterized by the following conditions:

$$a \geq b_1 + b_2 + b_3 \qquad \text{and} \qquad b_1 \geq b_2 \geq \ldots \geq b_l. \tag{3.7}$$

Clearly, those are the (non-strict) period conditions (3.1).

In the argumentation below we can suppress the vanishing coefficients b_i. Thus we may assume that $b_l \geq 1$. Alternatively, we can think that l is the index of the last non-vanishing b_i.

Denote by b^* the average value of the first three b_i: $b^* := (b_1 + b_2 + b_3)/3$. Then $b^* \leq a/3$, $b^* \geq b_i$ for $i \geq 3$ and

$$\textstyle\sum_i b_i \leq l \cdot b^*.$$

Since $\sum_i b_i = 3a + 1$, we get $3a + 1 \leq \frac{la}{3}$, or $l \geq 9 + \frac{3}{a}$. So we conclude that $l \geq 10$. Further, $3a + 1 \geq \sum_i b_i \geq l \geq 10$, and hence $a \geq 3$.

Let us recall the whole set of relations on the coefficients $a; b_i$. Those are positive integers satisfying period conditions (3.7), two more linear conditions

$$\textstyle\sum_i b_i = 3a + 1 \qquad \text{and} \qquad b_l \geq 0.$$

and the quadratic condition

$$\textstyle\sum_i b_i^2 = a^2 + 1.$$

Now, we shall perform some manipulations with coefficients b_i preserving the value of a and the *linear* conditions, trying to maximize the value of $\sum_i b_i^2$. To make our procedure uniform, we denote by $\boldsymbol{b} = (b_1, b_2, \ldots)$ any non-increasing sequences of integers such that $b_i = 0$ starting from some sufficiently large index i. In what follows those sequences $\boldsymbol{b}$ can be finite or infinite, this will play no role. In particular, for such sequences the values $\boldsymbol{b}^2 := \sum_i b_i^2$ and $tr(\boldsymbol{b}) := \sum_i b_i$ are well-defined.

Further, we assume that the linear conditions above are satisfied. In particular, $tr(\boldsymbol{b}) = \sum_i b_i = 3a+1$, $b_1 \geq b_2 \geq \cdots \geq b_i \geq 0$ and $b_1 + b_2 + b_3 \leq a$. For a given value of an integer a we want to find the maximal value of $\boldsymbol{b}^2$ on the set of such sequences satisfying those linear conditions. Notice that if this value is less or equal than a^2 then we obtain a contradiction with hypotheses of the theorem. This will give us the proof.

A trivial observation is that for $t > 0$ and $b_i \geq b_j$ the expression $(b_i + t)^2 + (b_j - t)^2$ is bigger than $b_i^2 + b_j^2$. This allows us to increase the value of $\boldsymbol{b}^2$ preserving $tr(\boldsymbol{b})$.

First, assume that $b_1 + b_2 + b_3 < a$. As before, l is the index of the last non-vanishing b_i. Then we can increase b_1 by some k and decrease b_l by the same k such that we get either $b_1 + b_2 + b_3 = a$ or $b_l = 0$. In the latter case the index of the last non-vanishing b_i becomes $l-1$. We do this operation until we get $b_1 + b_2 + b_3 = a$. Next, in the case $b_2 > b_3$ we increase b_1 by $b_2 - b_3$ and decrease b_2 by the same $b_2 - b_3$. In the obtained sequence $\boldsymbol{b}$ we have $b_1 + b_2 + b_3 = a$ and $b_2 = b_3$.

In the next step of the procedure we find the index $k \geq 4$ such that $b_3 = b_4 = \cdots = b_{k-1} > b_k$. If $k = l$ (the last non-vanishing b_i) or $k = l+1$ (this means that $b_l = b_3$ and $b_{l+1} = 0$) we do nothing. Otherwise, i.e. in the case $k < l$, we increase b_k and decrease b_l until we get either $b_k = b_3$ or $b_l = 0$. Then we try to repeat this procedure with new values of k or l.

If none of the above procedure is possible, we obtain the following sequence $\boldsymbol{b}$: It has $b_1 = a - 2b$, $b_2 = \cdots = b_{l-1} = b$, and $b \geq b_l > 0$. We denote b_l by c and write $c = \alpha b$. Then we get $0 < \alpha \leq 1$ and

$$3a + 1 = (a - 2b) + (l-2)b + \alpha b$$

which yields $l + \alpha = 4 + \frac{2a+1}{b}$. Now we can write

$$\begin{aligned}
\boldsymbol{b}^2 - a^2 &= (a-2b)^2 + (l-2)\, b^2 + \alpha^2 b^2 - a^2 \\
&= 4b^2 - 4ab + (l - 2 + \alpha)\, b^2 + (\alpha^2 - \alpha) b^2 \\
&= \left(6 + \frac{2a+1}{b}\right) b^2 - 4ab + (\alpha^2 - \alpha) b^2 \\
&= 6b^2 - 2ab + b + (\alpha^2 - \alpha) b^2
\end{aligned}$$

The latter is a quadratic polynomial in b with positive leading coefficient. It follows that its maximum value in any interval $[b_0, b_1]$ is taken at the end points. In our case this interval is $\left[1, \left[\frac{a}{3}\right]\right]$, where $\left[\frac{a}{3}\right]$ denotes the integer part of $\frac{a}{3}$. Let us notice that since $0 < \alpha \leq 1$ we get $0 \leq (\alpha - \alpha^2) \leq \frac{1}{4}$. At $b = b_0 = 1$ we get

$$\boldsymbol{b}^2 - a^2 = 6 - 2a + 1 - (\alpha - \alpha^2)\, b^2$$

Since $a \geq 3$, the latter expression is always ≤ 1 and the equality holds only in the case when $a = 3$ and all b_i are 1. This is the case when we have

$$[C] = 3L - \textstyle\sum_{i=1}^{10} E_i$$

Next, let us consider the value of the quadratic function at $b = \left[\frac{a}{3}\right]$ for different values of a modulo 3. For $a = 3m + q$ with $q = 1$ or $q = 2$ we get $b = m$ and

$$\begin{aligned}
6b^2 - 2ab + b - (\alpha - \alpha^2) b^2 &= 6m^2 - 6m^2 - 2qm + m - (\alpha - \alpha^2)\, m^2 \\
&= -m\,(2q-1) - (\alpha - \alpha^2)\, m^2
\end{aligned}$$

which is strictly negative. It follows that under assumption of the theorem we must have $a = 3m$ with an integer m. Moreover, the maximising sequence $\boldsymbol{b}$ must be of the form $b_1 = \cdots = b_9 = m$ and $b_{10} = 1$. □

Notice that from the proof of the theorem we obtain the following properties of the homology class $[C]$ of a symplectic (-1)-torus in a rational 4-manifold X:

Corollary 3.14. *Let (X, ω) be a rational symplectic 4-manifold and C a symplectically embedded (-1)-torus in X.*

1. *The integer m is defined uniquely.*
2. *In the case $m = 1$ the subgroup in $H_2(X, \mathbb{Z})$ generated by $L; E_1, \ldots, E_{10}$ is well-defined and independent of the choice of a basis used in the representation3.6. The classes $L; E_1, \ldots, E_{10}$ are defined up to the action of the reflection group generated by the reflections given by the classes $S'_{123}; S_{1,2}, \ldots, S_{9,10}$.*
3. *In the case $m > 1$ the class E_{10} and subgroup in $H_2(X, \mathbb{Z})$ generated by $L; E_1, \ldots, E_9$ are well-defined and independent of the choice of a basis used in the representation3.6. The classes $L; E_1, \ldots, E_9$ are defined up to the action of the reflection group generated by the reflections given by the classes $S'_{123}; S_{1,2}, \ldots, S_{8,9}$.*
4. *One can find exceptional symplectic spheres $E_1, \ldots, E_{10}$ in X which satisfy the properties of the* ***Theorem 3.13*** *and which intersect C transversally and positively.*
5. *Making a contraction of the exceptional sphere E_{10} we obtain a new rational symplectic 4-manifold (X', ω'). The torus C descends to X' as a symplectic torus of C' of self-intersection $[C']^2 = 0$.*

   Vice versa, let C' be a a symplectic torus of C' of self-intersection $[C']^2 = 0$ on a symplectic 4-manifold (X', ω'). Then making a symplectic blow-up of X' in a point p lying on C' we obtain a new symplectic 4-manifold (X, ω) and a symplectic torus of C on it with self-intersection $[C]^2 = -1$.

Proposition 3.15. *Let $l \geq 10$ and let X be a $\mathbb{CP}^2$ blown-up in l points. Let $L; E_1, \ldots, E_l$ be the geometric basis and let $\xi = \lambda[L] - \sum_i \mu_i[E_i]$ be the 2-cohomology class. Assume that $\mu_2 = \cdots = \mu_l = \mu$, and $\mu_1 = \lambda - 2\mu$. Assume additionally, that*

- $0 < \mu < \frac{2\lambda}{7}$ *if $l = 10$;*
- $0 < \mu < \frac{4\lambda}{l+3}$ *if $l \geq 11$.*

Then

$$\lambda = \mu_1 + \mu_2 + \mu_3, \quad \mu_1 > \mu_2 = \mu_3 = \cdots = \mu_l > 0,$$

the class ξ is represented by a symplectic form, and for every symplectic torus C in X with $C^2 = -1$ we have $\int_C \xi > 0$.

Proof. It is based on the proof of **Theorem 3.13**. The key observation is as follows. Recall that we write the homology class $[C]$ of the (-1)-torus in the form $[C] = aL - \sum_i b_i E_i$. Then we made the following transformations of the coefficients $a, b_1, \ldots, b_l$. If $b_i < b_{i+1}$ then we exchanged them setting $b'_i := b_{i+1}$ and $b'_{i+1} := b_i$. Next, if $a < b_1 + b_2 + b_3$, then we replaced

a, b_1, b_2, b_3 by $a' := a - d$, $b_1' := b_1 - d$, $b_2' := b_2 - d$, $b_3' := b_3 - d$ where $d := b_1 + b_2 + b_3 - a$. Let us denote the new class obtained by the first or second construction by $[C']$.

Now make the following observations: Both transformations of the coefficients can be made using a diffeomorphism f, i.e., $[C'] = f_*([C])$. Moreover, this isomorphism is the Dehn twist along the sphere $S_{i,i+1}$ in the first case, and S'_{123} in the second case. It follows that $c_1(X) \cdot [C'] = c_1(X) \cdot [C] = -1$ and $[C']^2 = [C]^2 = -1$.

The other observation is that $\xi \cdot [C'] \leq \xi \cdot [C] \leq 0$. Consequently, repeating the argumentation from the proof of **Theorem 3.13** we conclude that under the hypotheses of **Proposition 3.15** there exists an integer $m \geq 1$ such that for the class

$$[C] = 3mL - \sum_{i=1}^{9} mE_i - E_{10}$$

we have $\xi \cdot [C] \leq 0$. However, for ξ as in the hypotheses of the proposition we get $\xi \cdot [C] > 0$. This contradiction proves the proposition. □

4 Hilbert schemes of points on $\mathbb{CP}^2$ and special configurations of points.

Consider some X_l, which is the blowing up of $\mathbb{CP}^2$ at l points. Then we can define the sequence of contraction

$$X_l \xrightarrow{q_l} X_{l-1} \xrightarrow{q_{l-1}} X_{l-2} \to \cdots \xrightarrow{q_1} X_0 \cong \mathbb{CP}^2 \tag{4.1}$$

such that every map $X_i \xrightarrow{q_i} X_{i-1}$ is a contraction of a unique exceptional curve in the homology class E_i. Sequences as (4.1) we will call **blow-up sequences**.

We fix the integer l and now let us define a moduli space $\mathcal{X}_l$ of all blowings-ups of $\mathbb{CP}^2$ at l points as a set of all possible blow-up sequences (4.1) with fixed isomorphism $X_0 \cong \mathbb{CP}^2$. Forgetting the last contraction $X_l \xrightarrow{q_l} X_{l-1}$ we obtain the natural map between two moduli spaces $\mathcal{X}_l \xrightarrow{p_l} \mathcal{X}_{l-1}$.

For an explicit construction of the moduli space $\mathcal{X}_l$ see [4]. Now let us list some properties of the spaces $\mathcal{X}_l$.

Lemma 4.1. **(i)** *Consider the projection $p_{l+1} : \mathcal{X}_{l+1} \to \mathcal{X}_l$. This projection is a bundle with fiber $p_{l+1}^{-1}(\boldsymbol{x})$, which is rational 4-dimensional manifold isomorphic to $X_l(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, \ldots, x_l)$ is the sequence of blow up centers and $X_l(\boldsymbol{x})$ is the blow up of $\mathbb{CP}^2$ in centers $\boldsymbol{x} = (x_1, \ldots, x_l)$.*

(ii) *The group $\boldsymbol{PGl}(2, \mathbb{C})$ acts on $\mathbb{CP}^2$ and there is a extension of this action to the action of $\boldsymbol{PGl}(2, \mathbb{C})$ on $\mathcal{X}_l$ such that every projection $p_{l+1} : \mathcal{X}_{l+1} \to \mathcal{X}_l$ is equivariant.*

(iii) *There is a natural isomorphism between the quotient space $\boldsymbol{PGl}(2, \mathbb{C})\big/\mathcal{X}_l$ and the quotient space $\mathcal{J}^{int}(X)\big/Diff_{\blacksquare}(X)$.*

Remark 4.1. Notice that in general the action of $\boldsymbol{PGl}(2, \mathbb{C})$ on $\mathcal{X}_l$ is *not proper* and the quotient space $\boldsymbol{PGl}(2, \mathbb{C})\big\backslash\mathcal{X}_l$ is not Hausdorff.

Now let us explain how to interpret in terms of moduli spaces $\mathcal{X}_l$ and configurations $\boldsymbol{x}$ that the symplectic form ω and cohomology class ξ have type $\mathbf{D}_{l-1}$ or $\mathbf{E}_l$.

Consider the case when $\xi \in H^2(X, \mathbb{R})$ satisfy the non-strict period conditions. $R(X, \xi)$ is a system of positive root of Coxeter system $\mathcal{S}(X, \xi)$ (system that defines the reflection group $\Gamma_W(X, \xi)$). If $S \in R(X, \xi)$ then we know that S is an integer homology class, which is represented by a (-2)-sphere.

Definition 4.1. $\mathcal{X}_l(\xi)$ denotes the subset of that configuration of blow-up centers $\boldsymbol{x} \in \mathcal{X}_l$ for which there is no class $S \in R(X, \xi)$ represented by some curve $C \in X_l(\boldsymbol{x})$; $\mathcal{D}_l(\xi)$ is the following complement $\mathcal{X}_l \backslash \mathcal{X}_l(\xi)$.

Now we want to describe $\mathcal{X}_l(\xi)$ in the cases, when ξ has a type $\mathbf{D}_{l-1}$ or $\mathbf{E}_l$, more precisely, for the following simple root systems

(E) $\mathcal{S}(X, \xi) = \mathbf{E}_l = \{S'_{123}, S_{1,2}, S_{2,3}, \ldots, S_{l-1,l}\}$ and

(D) $\mathcal{S}(X, \xi) = \mathbf{D}_{l-1} = \{S'_{123}, S_{2,3}, S_{3,4}, \ldots, S_{l-1,l}\}$.

Remark 4.2. Notice that we consider the case, when system $\mathbf{E}_l$ has rank $l \leq 8$. Also, two systems $\mathbf{E}_5$ and $\mathbf{D}_5$ are isomorphic as abstract Coxeter systems, however, their realizations in $H_2(X, \mathbb{Z})$ are different. Consequently corresponding spaces are also not isomorphic, because they have even different dimensions.

Now describe the set of positive roots for the system $R(\mathrm{E}_l)$:

(S0) $S_{i,j} = E_i - E_j$ with $1 \leq i < j \leq l$;

(S1) $S'_{ijk} = L - (E_i + E_j + E_k)$ with $1 \leq i < j < k \leq l$;

(S2) $S''_{i_1,\dots,i_6} = 2L - \sum_j E_{i_j}$ with $1 \leq i_1 < i_2 < \ldots < i_6 \leq l$; here we must have $l = 6, 7, 8$;

(S3) $S'''_i = 3L - E_i - \sum_{i=1}^{8} E_i$, here we must have $l = 8$ and $1 \leq i \leq 8$;

And positive roots for the system $R(\mathrm{D}_{l-1})$ are the following:

(S'0) $S_{i,j} = E_i - E_j$ with $2 \leq i < j \leq l$;

(S'1) $S'_{1ij} = L - (E_1 + E_i + E_j)$ with $2 \leq i < j \leq l$.

Lemma 4.2. *The locus $\mathcal{D}_l(\xi)$ is a divisor whose irreducible components are indexed by the system of positive roots $R^+(X,\xi)$ of the system $\mathcal{S}(X,\xi)$.*

Recall that we have $4 \leq l \leq 9$ and $\mathcal{S}(X,\xi)$ is either E_l or D_{l-1}.

Lemma 4.3. **(i)** *The group $\mathbf{Aut}(\mathbb{CP}^2) = \mathbf{PGL}(3,\mathbb{C})$ acts on $\mathcal{X}_l$ and for each ξ the space $\mathcal{X}_l(\xi)$ is $\mathbf{PGL}(3,\mathbb{C})$-invariant.*

(ii) *There exists a natural map from the quotient $\mathcal{X}_l/\mathbf{PGL}(3,\mathbb{C})$ to the quotient $\mathcal{J}^{int}(X)/Diff_{\blacksquare}(X)$. There exists a closed algebraic set $\mathcal{A}_l \subset \mathcal{X}_l$ of complex codimension ≥ 2 such that pre-image of the set $\mathcal{J}(X,\xi)$ in $\mathcal{X}_l$ contains $\mathcal{X}_l(\xi)\backslash\mathcal{A}_l$.*

Corollary 4.4. *There exist a map from $\pi_1\big(\mathcal{X}_l(\xi)/\mathbf{PGL}(3,\mathbb{C})\big)$ to $\pi_0(Symp(X,\xi))$.*

By **Lemmas 3.11 and 3.12** the divisorial part of $\mathcal{D}^*_l(\xi)$ is $\mathcal{D}_l(\xi)$. □

Corollary 4.5. *There exist a map from $\pi_1\big(\mathcal{X}_l(\xi)/\mathbf{PGL}(3,\mathbb{C})\big)$ to $\pi_0(Symp(X,\xi))$.*

Proposition 4.6. *The fundamental group $\pi_1\big(\mathcal{X}^\circ_l(\xi, x^*_1)\big)$ is the pure braid group $PBr_{l-1}(S^2)$ of the sphere on $l-1$ strands.*

□

4.1 Presentation of the the fundamental group of the variety $\mathcal{X}_l(\xi)$ in the case D.

In the diagram

$$1 \to \pi_1(\mathfrak{M}(X,\xi)) \to \pi_1(\widehat{\mathfrak{M}}(X,\xi)) \to \Gamma_W(X,\xi) \to 1. \tag{4.2}$$

we have computed the first and the last terms for the case $\mathcal{S}(X,\xi) = \mathbf{D}_{l-1}$. Moreover, we have found presentations for those groups.

Recall that $\Gamma_W(X,\xi)$ is a reflection group of type $\mathbf{D}_{l-1}$ and the natural system of generators are reflections indexed by elements the system $\mathcal{S}(X,\xi)$. Moreover, those reflections can be represented by symplectomorphisms in $Symp(X,\omega)$. Namely, if S is a homology class from $\mathcal{S}(X,\xi)$, then this class is represented by an ω-Lagrangian sphere $\Sigma \subset X$, and the symplectic Dehn twist along Σ, denoted by T_Σ, is the desired symplectomorphism.

Let us notice that such the square T_Σ^2 of such a twist is smoothly isotopic to identity, i.e. $T_\Sigma^2 = 1$ in the group $\Gamma_W(X,\xi)$. In this way we obtain certain elements in the group $\pi_1(\mathfrak{M}(X,\xi))$. We are going to show that those elements generate the whole group $\pi_1(\mathfrak{M}(X,\xi))$, and find a natural systems of relations between those generators.

Claim 4.7. *Let $\Sigma \subset X$ be an ω-Lagrangian sphere with the homology class S and T_Σ the corresponding symplectic Dehn twist. Then the symplectic isotopy class $[T_\Sigma^2]$ in the group $\pi_0(Symp(X,\omega))$ is the image of the loop around certain component $\mathcal{D}(S)$ of the divisor $\mathcal{D}_l(\xi)$. Moreover, the class S lies in the root system $R^+(\xi)$, and the component $\mathcal{D}(S)$ is the locus of constellations of points $x_1, \ldots, x_l$ as in Lemma 4.2.*

5 Generators and relations in the case of type D.

5.1 Some preliminaries.

Let us recall main results about the group $\widehat{\pi}_1\big(\widehat{\mathcal{M}}_l(\mathsf{D})\big)$.

- The constellation we consider has rank $l-1$. The corresponding homology classes $\mathbf{S}'_{123}; \mathbf{S}_{12}, \dots, \mathbf{S}_{l-1,l}$ in X compose a constellation of the type D_{l-1}. Moreover, those classes can be represented by Lagrangian spheres $S'_{123}; S_{12}, \dots, S_{l-1,l}$.
- The whole group is generated by the elements $s'_{123}; s_{23}, \dots s_{l-1,l}$ which are symplectic Dehn twists along the Lagrangian spheres $S'_{123}; S_{12}, \dots, S_{l-1,l}$. Moreover, those generators satisfy the braid relations corresponding to the diagram D_{l-1}. Consequently, the group $\widehat{\pi}_1\big(\widehat{\mathcal{M}}_l(\mathsf{D})\big)$ is a quotient of the braid group $\mathsf{Br}(\mathsf{D}_{l-1})$.
- The group $\mathsf{Br}(\mathsf{D}_{l-1})$ acts on the homology group $H_2(X,\mathbb{Z})$ as the reflection group $\mathsf{W}(\mathsf{D}_{l-1})$. The kernel of this action is the pure braid group $\mathsf{PBr}(\mathsf{D}_{l-1})$.
- The pure braid group $\mathsf{PBr}(\mathsf{D}_{l-1})$ is generated by the squares of Dehn twists $(s'_{1ij})^2$ and s^2_{ij} along the Lagrangian spheres in the classes S'_{1ij} and S_{ij} where $2 \le i < j \le l$. The homology classes S'_{1ij} and S_{ij} compose the system of positive roots of type D_{l-1}. On the other hand, those squares of Dehn twists $(s'_{1ij})^2$ and s^2_{ij} are represented in $\pi_1\big(\widehat{\mathcal{M}}_l(\mathsf{D})\big)$ by paths around the divisors $\mathcal{D}'_{123}; \mathcal{D}_{23}, \dots, \mathcal{D}_{l-1,l}$.
- There exists a (natural) epimorphism of the groups $\mathsf{Br}(\mathsf{D}_{l-1}) \to \mathsf{Br}(\mathsf{A}_{l-2})$ defined by adding the relation $s'_{123} = s_{23}$ to the set of standard defining relations of $\mathsf{Br}(\mathsf{D}_{l-1})$. We denote this epimorphism by ρ. It induces two other epimorphisms $\rho : \mathsf{W}(\mathsf{D}_{l-1}) \to \mathsf{W}(\mathsf{A}_{l-2})$ and $\rho : \mathsf{PBr}(\mathsf{D}_{l-1}) \to \mathsf{PBr}(\mathsf{A}_{l-2})$. Recall that $\mathsf{W}(\mathsf{A}_{l-2}) = \mathsf{Sym}_{l-1}$ and $\mathsf{PBr}(\mathsf{A}_{l-2}) = \mathsf{Br}_{l-1}$ are the standard symmetric and braid groups. The group Sym_{l-1} is naturally realized as the permutation group of the classes $\mathbf{E}_2, \dots, \mathbf{E}_l$ from the geometric(?) basis of X.
- The homomorphism from $\mathsf{PBr}(\mathsf{D}_{l-1})$ to $\pi_1\big(\widehat{\mathcal{M}}_l(\mathsf{D})\big)$ factorizes through the epimorphism $\rho : \mathsf{PBr}(\mathsf{D}_{l-1}) \to \mathsf{PBr}(\mathsf{A}_{l-2})$.
- We have natural homomorphisms from the usual braid groups (full and pure) to the corresponding braid groups of the sphere:

$$\sigma : \mathsf{Br}_{l-1} \to \mathsf{Br}_{l-1}(S^2) \ \text{ and } \ \sigma : \mathsf{PBr}_{l-1} \to \mathsf{PBr}_{l-1}(S^2).$$

- The center of the braid group Br_{l-1} is a free abelian group $\mathbb{Z}$ generated by the square of the **Garside element** Δ^2_{l-1}. It lies in the pure braid group PBr_{l-1}. We use the same notation for its image in the braid group of the sphere.

Let us explain now the goal of this section. We want to do the following

1. Show that in the case D the symplectic mapping class group $SMap(X_l, \mathsf{D})$ is isomorphic to the group $\widehat{\pi}_1\big(\widehat{\mathcal{M}}_l(\mathsf{D})\big)$.
2. Show that latter group is isomorphic to the image of the braid group $\mathsf{Br}(\mathsf{D}_{l-1})$ in the product

$$\mathsf{W}(\mathsf{D}_{l-1}) \times \mathsf{Br}_{l-1}(S^2)/\mathbb{Z}\langle \Delta^2_{l-1} \rangle.$$

3. Find the arising "geometric presentation" of the symplectic mapping class group $SMap(X_l, \mathbf{D})$. Namely, we want to show that this group is the quotient of the braid group $\mathsf{Br}(\mathbf{D}_{l-1})$ arising by adding the following relations:
   - $(s'_{123})^2 = s_{23}^2$;
   - $\varDelta_{l-1}^2 = 1$;
   - $\varDelta_{l-2}^2 = 1$, where $\varDelta_{l-1}^2$ is the square of the Garside element of the subgroup Br_{l-2} of Br_{l-1} generated by the subset $\{s_{34}, \dots, s_{l-1,l}\}$.

5.2 Braid group and mapping class group of the sphere.

Recall the definition of the **braid group of the sphere** $\mathsf{Br}_l(S^2)$. This is the fundamental group of the configuration space which is constructed in the same manner as in the case of usual braid group:

- Take the l-fold product $S^2 \times \cdots \times S^2 = \big(S^2\big)^l$;
- Remove the diagonal set Δ which is the union of divisors $\Delta_{ij} := \{z_i = z_j\}$;
- Take the quotient of $\big(S^2\big)^l \backslash \Delta$ by the symmetric group Sym_l. This is the configuration space of the braid group of the sphere $\mathsf{Br}_l(S^2)$.
- The latter action of Sym_l on $\big(S^2\big)^l \backslash \Delta$ is free. It follows that we get a surjective homomorphism which gives rise the following extension of groups:

  $$1 \to \mathsf{PBr}_l(S^2) \to \mathsf{Br}_l(S^2) \to \mathsf{Sym}_l \to 1.$$

  Moreover, the kernel $\mathsf{PBr}_l(S^2)$, called the **pure braid group of the sphere**, is the fundamental group of the complement $\big(S^2\big)^l \backslash \Delta$.

The "standard" (geometric?) presentation of the spherical braid group is as follows (see [10, 11]): We take the standard presentation of the usual braid group Br_l with the generators $s_1, \dots, s_{l-1}$ and add the **spherical relation**

$$s_1 s_2 \dots s_{l-2} s_{l-1}\, s_{l-1} s_{l-2} \dots s_2 s_1 = 1.$$

Notice that the spherical relation can be also written in the form

$$s_{l-1} s_{l-2} \dots s_2 s_1\, s_1 s_2 \dots s_{l-2} s_{l-1} \; = 1$$

or in terms of Garside elements as

$$\varDelta_l^2 = \varDelta_{l-1}^2.$$

Here $\varDelta_{l-1}$ is the Garside element of the subgroup Br_{l-1} generated by $s_1, \dots, s_{l-2}$.

Next, we want to find a presentation of the mapping class group of the sphere S^2 with l marked points. We denote this group by $Map(S^2, l)$ in the case when the points are (allowed to be) permuted, and by $Map(S^2, l!)$ when points are fixed. Recall that in the case on the disc D or the plane $\mathbb{R}^2$ we have natural isomorphisms

$$Map(\mathbb{R}^2, l) = Map(D, l) \cong \mathsf{Br}_l \quad \text{and} \quad Map(\mathbb{R}^2, l!) = Map(D, l!) \cong \mathsf{PBr}_l,$$

see e.g. [14]. In the case of the braid group of the sphere the situation is almost the same.

Theorem 5.1. *Let $l \geq 3$.*

(1) ([12, 49]) *In the braid group of the sphere one has the relation $\mathit{\Delta}_l^4 = 1$.*

(2) *The center of the group $\mathsf{Br}_l(S^2)$ has order 2 and is generated by $\mathit{\Delta}_l^2$*

(3) *There are natural isomorphisms of the mapping class groups of the sphere*

$$Map(S^2, l) \cong \mathsf{Br}_l(S^2, l)/\mathbb{Z}_2\langle \mathit{\Delta}_l^2 \rangle \quad and \quad Map(S^2, l!) \cong \mathsf{PBr}_l(S^2, l)/\mathbb{Z}_2\langle \mathit{\Delta}_l^2 \rangle. \tag{5.1}$$

(4) *In particular, the mapping class group of the sphere with l punctures $Map(S^2, l)$ is obtained from the braid group Br_l by adding two relations $\mathit{\Delta}_l^2 = 1$ and $\mathit{\Delta}_{l-1}^2 = 1$.*

Remark 5.1. Since $\mathit{\Delta}_l^2 = \mathit{\Delta}_{l-1}^2$ in the braid group of sphere, the relation $\mathit{\Delta}_l^4 = 1$ is equivalent to the relation $\mathit{\Delta}_{l-1}^4 = 1$. The key property of the relation $\mathit{\Delta}_l^4 = 1$ is that it follows from the spherical relation $\mathit{\Delta}_l^2 = \mathit{\Delta}_{l-1}^2$ (and other braid relations, but nothing more).

In the papers [12, 49] this relation is written in the form $(s_1 \ldots s_{l-1})^{2l} = 1$.

In [49] Lee van Buskirk has shown that for $l \geq 3$ $\mathit{\Delta}_l^2$ is a non-trivial element even in the group $\mathsf{Br}_l(\mathbb{RP}^2)$ (under the natural homomorphism $\mathsf{Br}_l(S^2) \to \mathsf{Br}_l(\mathbb{RP}^2)$).

Proof. First we recall (and show) that $(s_1 \ldots s_{l-1})^l = \mathit{\Delta}_l^2$ in any braid group Br_l. This explains the equivalence of the two forms $(s_1 \ldots s_{l-1})^{2l} = 1$ and $\mathit{\Delta}_l^4 = 1$.

Let $\mathcal{S} = \{s_1, \ldots, s_l\}$ be any irreducible Coxeter of compact type. Recall that the latter means that the corresponding Coxeter group $\mathsf{W}(\mathcal{S})$ is finite. In this case $\mathsf{W}(\mathcal{S})$ has the unique element of the maximal length, called the **longest element** and denoted by $w_\circ$ or by $w_\circ(\mathcal{S})$. Its primitive lift to the braid group $\mathsf{Br}(\mathcal{S})$ is called the **Garside element** and denoted by $\mathit{\Delta}$ or by $\mathit{\Delta}(\mathcal{S})$. Next, let $\varkappa = \prod_{i=1}^l s_i$ be a product of the elements of $\mathcal{S}$, taken in any order. (Notice that each generator s_i appears exactly once.) Such an element $\varkappa$ is called **Coxeter element** of the group $\mathsf{W}(\mathcal{S})$, and its order is called the **Coxeter number.** We denote the Coxeter number by h or by $h_{\mathcal{S}}$. It is known that all Coxeter elements (corresponding to all possible orders in the system $\mathcal{S}$) are conjugated in $\mathsf{W}(\mathcal{S})$. It follows that the Coxeter number is well defined.

Let $\hat{\varkappa}$ be the primitive lift of the Coxeter element to the braid group $\mathsf{Br}(\mathcal{S})$. Consider the power $\hat{\varkappa}^h$. It has two properties: first, it lies in the monoid $\mathsf{Br}^+(\mathcal{S})$ of positive braids, and second, its projection to $\mathsf{W}(\mathcal{S})$ is trivial. One can show that, under the assumption above about the irreducibility of the system $\mathcal{S}$, the power $\hat{\varkappa}^h$ is a power of $\mathit{\Delta}^2(\mathcal{S})$, i.e., an even power of $\mathit{\Delta}(\mathcal{S})$. Another fact is that this must be the minimal possible power, which means the relation $\hat{\varkappa}^h = \mathit{\Delta}^2(\mathcal{S})$.

The standard Coxeter element in the symmetric group Sym_l is $\varkappa = s_1 s_2 \ldots s_{l-1}$ which is a cycle of the maximal length $(l, 1, 2, 3, \ldots, l-1)$ and its order in l. This gives us the relation $(s_1 \ldots s_{l-1})^l = \mathit{\Delta}_l^2$ in the braid group Br_l.

Next, let us denote for a moment $\widehat{\mathcal{X}}_l := \big(S^2)^l \backslash \Delta$ and $\mathcal{X}_l := \big((S^2)^l \backslash \Delta\big)/\mathsf{Sym}_l$ the configuration spaces for the braid group and the pure braid group of the sphere. Those spaces should be not confused with the configuration spaces of points on $\mathbb{CP}^2$ (even if we use the same notation). Consider the natural action of the diffeomorphism group $Diff_+(S^2)$ on those spaces. These actions are transitive, and the stabilisers of a configuration $\boldsymbol{x} = \{x_1, \ldots, x_l\}$ are the groups $Diff_+(S^2, \boldsymbol{x}!)$ and $Diff_+(S^2, \boldsymbol{x})$. Here as above the notation $\boldsymbol{x}!$ means that the points are not permuted. This gives us the principle bundles:

$$Diff_+(S^2, \boldsymbol{x}!) \xhookrightarrow{\ i\ } Diff_+(S^2) \longrightarrow \widehat{\mathcal{X}}_l$$

$$Diff_+(S^2, \boldsymbol{x}) \xhookrightarrow{\ i\ } Diff_+(S^2) \longrightarrow \mathcal{X}_l$$

The induced long exact sequences of homotopy groups are

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ \pi_1 \mathit{Diff}_+(S^2, \boldsymbol{x}!) & \longrightarrow & \pi_1 \mathit{Diff}_+(S^2) & \longrightarrow & \pi_1 \widehat{\mathcal{X}}_l & \xrightarrow{\partial} & \pi_0 \mathit{Diff}_+(S^2, \boldsymbol{x}!) & \longrightarrow & \pi_0 \mathit{Diff}_+(S^2) \\ & & \downarrow= & & \downarrow & & & & \\ \pi_1 \mathit{Diff}_+(S^2, \boldsymbol{x}) & \longrightarrow & \pi_1 \mathit{Diff}_+(S^2) & \longrightarrow & \pi_1 \mathcal{X}_l & \xrightarrow{\partial} & \pi_0 \mathit{Diff}_+(S^2, \boldsymbol{x}) & \longrightarrow & \pi_0 \mathit{Diff}_+(S^2) \\ & & & & \downarrow & & & & \\ & & & & \mathsf{Sym}_l & & & & \end{array}$$

Taking into account the "values" of the groups in the diagram we get

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathsf{PBr}_l(S^2) & \xrightarrow{\partial} & \mathit{Map}(S^2, \boldsymbol{x}!) & \longrightarrow & 1 \\ & & \downarrow= & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathsf{Br}_l(S^2) & \xrightarrow{\partial} & \mathit{Map}(S^2, \boldsymbol{x}) & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathsf{Sym}_l & \xrightarrow{=} & \mathsf{Sym}_l & & \end{array}$$

This implies the assertion of the theorem. □

Let us explain why the element Δ_l^2 has order 2 in $\mathsf{Br}_l(S^2)$ and is trivial in $\mathit{Map}(S^2, l)$. Imagine / assume that our points $\boldsymbol{x}$ lie in the lower hemisphere S^2_- , and λ be the equator of S^2. Recall that in the braid group $\mathsf{Br}_l \cong \mathit{Map}(D, l) \cong \mathit{Map}(S^2_-, l)$ the element Δ_l^2 is represented by the Dehn along the boundary of the disc. To the group $\mathit{Diff}_+(S^2)$ this twist is lifted as follows. We make the full turn of the lower hemisphere, leaving the upper hemisphere unmoved. This will give us the Dehn twist near equator. However, since all points lie in the lower hemisphere, the movement of the upper hemisphere plays no role, and we can use any diffeomorphism there. So we get the same braid Δ_l^2 if we apply the full rotation of the sphere. Notice, that we have a homotopy equivalence $\mathit{Diff}_+(S^2) \simeq \boldsymbol{SO}(3)$ induced by the embedding $\boldsymbol{SO}(3) \subset \mathit{Diff}_+(S^2)$ and hence $\pi_1 \mathit{Diff}_+(S^2) = \mathbb{Z}_2$ generated by the full rotation of the sphere.

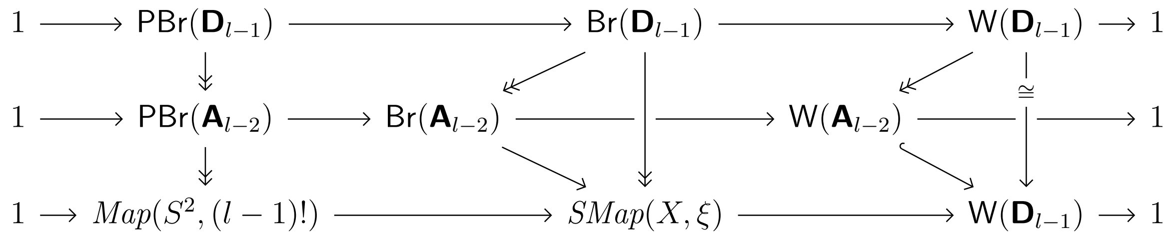
5.3 Relations in the case D.

Recall what we know about generators and relations of the group $\mathit{SMap}(X, \xi)$ in the case **D**:

- The group $\mathit{SMap}(X, \xi)$ contains Dehn twists T_S along some Lagrangian spheres in appropriate homology classes; the group generated by all such twists admit the following set of generators: $s'_{123}; s_{2,3}, s_{3,4}, \dots, s_{l-1,l}$. They are Dehn twists along the spheres in the classes $S'_{123}; S_{2,3}, S_{3,4}, \dots, S_{l-1,l}$.
- The incidences between the classes $S'_{123}; S_{2,3}, S_{3,4}, \dots, S_{l-1,l}$ is the same as in the system of simple roots of **D** and rank $l-1$; In particular, the group generated by Dehn twists along Lagrangian spheres is a quotient group of the braid group $\mathsf{Br}(\mathbf{D}_{l-1})$.

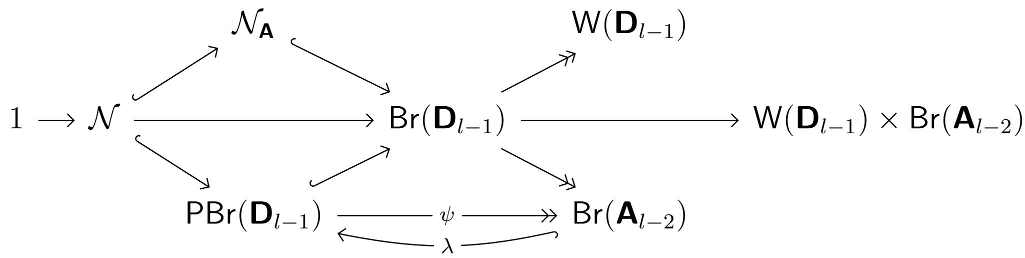
- There exists a homomorphism from the group $\mathsf{Br}(\mathbf{D}_{l-1})$ to the group $\mathsf{Br}(\mathbf{A}_{l-2})$ which is obtained by adding an additional relation $s'_{123} = s_{2,3}$. Recall also that we have a natural identification $\mathsf{Br}(\mathbf{A}_{l-2}) = \mathsf{Br}_{l-1}$. The group Br_{l-1} can be realised either as the subgroup of $\mathsf{Br}(\mathbf{D}_{l-1})$ generated by $s'_{123}; s_{3,4}, \dots, s_{l-1,l}$ (generator $s_{2,3}$ is excluded), or as the generated by $s_{2,3}s_{3,4}, \dots, s_{l-1,l}$ (generator s'_{123} is excluded).
- The image of the group $\mathit{SMap}(X,\xi)$ in the diffeotopy group $\pi_0(\mathit{Diff}_+(X))$ is the group $\Gamma_W(X,\xi)$, isomorphic to the Coxeter group $\mathsf{W}(\mathbf{D}_{l-1})$ and the elements $s'_{123}; s_{2,3}, s_{3,4}, \dots, s_{l-1,l}$ form the Coxeter system of the standard generators of this group.
- Finally, the kernel of the homomorphism $\mathit{SMap}(X,\xi) \to \pi_0(\mathit{Diff}_+(X))$ is isomorphic to the mapping class group of the sphere with $(l-1)$ non-permuted marked points. In our notation above $\mathit{Map}(S^2,(l-1)!)$.

Altogether, we get the diagram which summarises relevant groups.



We see that the system $\mathbf{D}_{l-1} = \{s'_{123}; s_{2,3}, s_{3,4}, \dots, s_{l-1,l}\}$ generates the group $\mathit{SMap}(X,\xi)$ and the generators in this system satisfy the braid relations of type $\mathbf{D}_{l-1}$. This fact is encoded in the epimorphism $\mathsf{Br}(\mathbf{D}_{l-1}) \twoheadrightarrow \mathit{SMap}(X,\xi)$. We get the desired presentation of the group $\mathit{SMap}(X,\xi)$ if we find remaining relations between the generators $s'_{123}; s_{2,3}, s_{3,4}, \dots, s_{l-1,l}$.

Our problem can be described using yet another commutative diagram.



(5.2)

Namely, we must add the following relation: If an element $x \in \mathsf{Br}(\mathbf{D}_{l-1})$ lies in the pure braid group $\mathsf{PBr}(\mathbf{D}_{l-1})$, and if its image $\psi(x)$ in $\mathsf{Br}(\mathbf{A}_{l-2})$ is trivial, then the element x is trivial in $\mathit{SMap}(X,\xi)$. This property could be restated as follows: We impose the condition $x \in \mathsf{Br}(\mathbf{D}_{l-1})$ *is trivial in* $\mathit{SMap}(X,\xi)$ *if it is trivial in* $\mathsf{Br}(\mathbf{A}_{l-2})$ only on elements x lying in the subgroup. The subgroup of such elements is the intersection of kernels of the homomorphisms $\mathsf{Br}(\mathbf{D}_{l-1}) \to \mathsf{W}(\mathbf{D}_{l-1})$ and $\mathsf{Br}(\mathbf{D}_{l-1}) \to \mathsf{Br}(\mathbf{A}_{l-2})$. The first kernel is $\mathsf{PBr}(\mathbf{D}_{l-1})$, and the second is denoted by $\mathcal{N}_{\mathbf{A}}$ as on the diagram (5.2). This intersection subgroup, denoted by $\mathcal{N}$ is the kernel of the homomorphism $\mathsf{Br}(\mathbf{D}_{l-1}) \to \mathsf{W}(\mathbf{D}_{l-1}) \times \mathsf{Br}(\mathbf{A}_{l-2})$.

Next, observe that $\mathcal{N}$ in a normal subgroup in $\mathsf{Br}(\mathbf{D}_{l-1})$. So finding a set of relations means to find a set of normal generators for the subgroup $\mathcal{N}$ in the ambient group.

To simplify the notation we redenote the generators $s'_{123}; s_{2,3}, s_{3,4}, \dots, s_{l-1,l}$ setting $a_0 =: s'_{123}$ and $a_1 =: s_{2,3}, a_2 =: s_{3,4}, \dots, a_{l-2} =: s_{l-1,l}$. Then the natural normal generator for $\mathcal{N}_{\mathbf{A}}$ is

$a_1a_0^{-1}$, since it corresponds to the relation $s'_{123} = s_{2,3}$ transforming the group $\mathsf{Br}(\mathbf{D}_{l-1})$ into $\mathsf{Br}(\mathbf{A}_{l-2})$. Next, recall that the epimorphism $\psi : \mathsf{Br}(\mathbf{D}_{l-1}) \to \mathsf{Br}(\mathbf{A}_{l-2})$ admits a natural splitting $\lambda : \mathsf{Br}(\mathbf{A}_{l-2}) \to \mathsf{Br}(\mathbf{D}_{l-1})$ which is the natural embedding induced by the inclusion of the sets of generators $\mathbf{A}_{l-2} \subset \mathbf{D}_{l-1}$.

We use the splitting λ to find a "normal form" for elements in $\mathcal{N}_{\mathbf{A}}$. Let us use the notation

$$x * y := xyx^{-1}$$

for conjugation of elements in a group.

Lemma 5.2. *Every element x in $\mathcal{N}_{\mathbf{A}}$ can be represented as a product*

$$x = \left(y_1 * (a_0a_1^{-1})^{\epsilon_1}\right) \cdot \ldots \cdot \left(y_n * (a_0a_1^{-1})^{\epsilon_n}\right) = \prod_i y_i * (a_0a_1^{-1})^{\epsilon_i} \tag{5.3}$$

with some elements y_i in the subgroup $\mathsf{Br}(\mathbf{A}_{l-2})$ and some signs $\epsilon_i = \pm 1$.

Proof. We use the rewriting procedure from the **Reidemeister-Schreier algorithm** for finding a presentation of a subgroup H of an ambient group G given by a presentation $G = \langle \mathcal{X} | \mathcal{R} \rangle$ with a set of generators $\mathcal{X}$ and a set of relations $\mathcal{R}$, see [8, 28, 29] for a description of the algorithm. For an element x in the group $\mathsf{Br}(\mathbf{D}_{l-1})$ we denote by $\bar{x}$ the image $\lambda(\psi(x))$. It lies in the subgroup $\mathsf{Br}(\mathbf{A}_{l-2})$. Moreover, we have three easy properties. The first one is that $\bar{x} = x$ if x lies in $\mathsf{Br}(\mathbf{A}_{l-2})$. The second is relation $\overline{x_1 \cdot x_2} = \bar{x}_1 \cdot \bar{x}_2$, which just means that $\lambda \circ \psi$ is a homomorphism. Finally, a product $z_1 \cdot z_2$ of two factorisations of the form (5.3) is again of the form (5.3).

Claim. *Every element $x \in \mathsf{Br}(\mathbf{D}_{l-1})$ admits a decomposition*

$$x = z \cdot \bar{x}$$

where z is a product of the form (5.3).

We prove this using the induction by the length of words representing a given element x. Since such a presentation surely exists for the unit 1, we can use it as the base of the induction. So we must prove the step of the induction. For this purpose it is sufficient to show, that if a given x admits a decomposition $x = z \cdot \bar{x}$, then so does the product $x_1 = x \cdot a_i^{\epsilon}$ for every generator a_i and every sign $\epsilon = \pm 1$. Further, if a generator a_i is not a_0, then it lies in the subgroup $\mathsf{Br}(\mathbf{A}_{l-2})$, and then $x_1 = z \cdot (x a_i^{\epsilon})$ will be the desired decomposition.

So the only case to consider is when the generator a_i is a_0. In the case $\epsilon = +1$ we get

$$x \cdot a_0 = z \cdot \bar{x} \cdot a_0 = z \cdot (\bar{x} \cdot a_0 a_1^{-1} \cdot \bar{x}^{-1}) \cdot \bar{x} \cdot a_1 = \underline{z \cdot (\bar{x} * (a_0 a_1^{-1}))} \cdot (\bar{x} \cdot a_1).$$

The underlined part is a new part z_1, whereas

$$\overline{x \cdot a_0} = \bar{x} \cdot \bar{a}_0 = \bar{x} \cdot a_1.$$

So $x \cdot a_0$ admits a decomposition of the desired form. The product $x \cdot a_0^{-1}$ is treated similarly. This finishes the proof of the claim.

Now, applying the claim to elements x from $\mathcal{N}_{\mathbf{A}}$ we conclude the assertion of the lemma. □

Our next step is related to the following diagram with exact rows.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{N}_{\mathsf{A}} & \underset{\psi_{\mathcal{N}}}{\overset{\lambda_{\mathcal{N}}}{\rightleftarrows}} & \mathbb{Z}_2^{l-2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathsf{PBr}(\mathbf{D}_{l-1}) & \longrightarrow & \mathsf{Br}(\mathbf{D}_{l-1}) & \longrightarrow & \mathsf{W}(\mathbf{D}_{l-1}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \psi_{\mathsf{W}} \downarrow \uparrow \lambda_{\mathsf{W}} & & \\
1 & \longrightarrow & \mathsf{PBr}(\mathbf{A}_{l-2}) & \longrightarrow & \mathsf{Br}(\mathbf{A}_{l-2}) & \underset{\psi_{\mathsf{Br}}}{\overset{\lambda_{\mathsf{Br}}}{\rightleftarrows}} & \mathsf{W}(\mathbf{A}_{l-2}) & \longrightarrow & 1
\end{array} \tag{5.4}$$

The homomorphisms ψ_{W} and λ_{W} on the diagram have the meaning similar to the one for the diagram (5.2): λ_{W} is a splitting homomorphism, and the group $\mathsf{W}(\mathbf{D}_{l-1})$ is a semi-direct product $\mathsf{W}(\mathbf{A}_{l-2}) \ltimes \mathbb{Z}_2^{l-2}$. The map $\lambda_{\mathcal{N}}$ is *not a homomorphism*, it will be constructed below.

In particular, we obtain a natural action of $\mathsf{W}(\mathbf{A}_{l-2})$ on $\mathbb{Z}_2^{l-2}$ by conjugation. We denote this action by $p * u$ with $p \in \mathsf{W}(\mathbf{A}_{l-2})$ and $u \in \mathbb{Z}_2^{l-2}$.

It follows from the definition of the group $\mathcal{N}$ that this group can be defined as the kernel of the natural homomorphism $\mathcal{N}_{\mathsf{A}} \to \mathsf{W}(\mathbf{D}_{l-1})$. However, since the composition $\mathcal{N}_{\mathsf{A}} \to \mathsf{W}(\mathbf{D}_{l-1}) \to \mathsf{W}(\mathbf{A}_{l-2})$ is zero, the image of $\mathcal{N}_{\mathsf{A}}$ in $\mathsf{W}(\mathbf{D}_{l-1})$ lies in the kernel of the homomorphism $\mathsf{W}(\mathbf{D}_{l-1}) \to \mathsf{W}(\mathbf{A}_{l-2})$. The latter kernel is isomorphic to $\mathbb{Z}_2^{l-2}$. The description of the groups in the short exact sequence

$$1 \to \mathbb{Z}_2^{l-2} \to \mathsf{W}(\mathbf{D}_{l-1}) \to \mathsf{W}(\mathbf{A}_{l-2}) \to 1$$

is as follows. We realise the group $\mathsf{W}(\mathbf{D}_{l-1})$ as the reflection group in the space $\mathbb{R}^{l-1}$ with the coordinates $x_1, \ldots, x_{l-1}$ which permutes the coordinates x_i and inverts the even number of them. Each generator a_i, $i = 1, \ldots, l-2$ is a reflection across the hyperplane given by the equation $x_i - x_{i+1} = 0$, so a_i exchanges the coordinate x_i with x_{i+1}. The generator a_0 is a reflection across the hyperplane given by the equation $x_1 + x_2 = 0$, so a_0 replaces the pair (x_1, x_2) by $(-x_2, -x_1)$. So if permutations of axes are not allowed, only the inversion of coordinates remain. This group is our $\mathbb{Z}_2^{l-2}$.

We use the multiplicative notation for the group operation in $\mathbb{Z}_2^{l-2}$. Let us introduce the following elements of the group $\mathbb{Z}_2^{l-2}$ acting on $\mathbb{R}^{l-1}$ in the way described above. By r_{ij} we denote the involution inverting two axes: x_i and x_j. We have clearly two sorts of relations: $r_{ij}^2 = 1$ and $r_{ij} \cdot r_{jk} = r_{ik}$ if $i \neq j \neq k \neq i$. Notice also that r_{12} is the image of the generator $a_0 a_1^{-1}$ in $\mathbb{Z}_2^{l-2}$. To make our notation uniform we set $r_{ii} := 1$ and $r_{ji} = r_{ij}$. As a set of generators of $\mathbb{Z}_2^{l-2}$ we choose $r_{12}, r_{13}, \ldots, r_{1,l-1}$. This is a basis of $\mathbb{Z}_2^{l-2}$ as $\mathbb{Z}_2$-vector space.

Now let us find a set of generators applying a modified version of the Reidemeister-Schreier algorithm.

Theorem 5.3. *The group SMap(X, ξ) of type* $\mathbf{D}$ *is the quotient of the group* $\mathsf{Br}(\mathbf{D}_{l-1})$ *obtained by adding the relations*

(R1) $[b, a_0 a_1^{-1}] = 1$ *for every b in the pure braid group* $\mathsf{PBr}(\mathbf{A}_{l-2})$*;*

(R2) $(a_0 a_1^{-1})^2 = 1$.

Proof. An equivalent assertion is that the subgroup $\mathcal{N}_{\mathsf{A}}$ in the group $\mathsf{Br}(\mathbf{D}_{l-1})$ admits the following system of normal generators:

(r1) $[b, a_0a_1^{-1}]$ for every b in the pure braid group $\mathsf{PBr}(\mathbf{A}_{l-2})$;

(r2) $(a_0a_1^{-1})^2$.

First, we must check that the relations in the list hold. This means vanishing of corresponding elements in both groups $\mathsf{W}(\mathbf{D}_{l-1})$ and $\mathsf{Br}(\mathbf{A}_{l-2})$. This can be done easily. Indeed, by the very definition of the pure braid group of any Coxeter system $\mathcal{S}$, b lies in the pure braid group iff its image in the Coxeter group $\mathsf{W}(\mathcal{S})$ vanishes. Then the commutator $[b, x]$ will also vanish. Similarly, since $a_0a_1^{-1}$ vanishes in $\mathsf{Br}(\mathbf{A}_{l-2})$ the commutator $[b, a_0a_1^{-1}]$ must also vanish. The vanishing of $(a_0a_1^{-1})^2$ in both groups is also an easy fact.

In the proof of the theorem we shall need some more relations which are consequences of the relations (R2), (R3). Let us make the notation

(R0) The braid relations between the generators $a_0; a_1, \dots, a_{l-2}$.

The next relation we need is a consequence of the braid relations. Let $b \in \mathsf{Br}(\mathbf{A}_{l-2})$ be a positive *primitive braid*, i.e., a braid whose length in $\mathsf{Br}(\mathbf{A}_{l-2})$ and in $\mathsf{W}(\mathbf{A}_{l-2})$ is the same. In terms of the arrows from the diagram (5.4) this means that $b = \lambda_{\mathsf{Br}}(\psi_{\mathsf{Br}}(b))$. Let us denote by $\bar{b}$ the permutation of the set $\{1, \dots, l-1\}$ defined by b, i.e., $\bar{b} = \psi_{\mathsf{Br}}(b)$. Assume that $\bar{b}$ commutes with r_{12} in the group $\mathsf{W}(\mathbf{D}_{l-1})$. We claim that in this case b commutes with $a_1a_0^{-1}$ in the group $\mathsf{Br}(\mathbf{D}_{l-1})$. The commutativity $[\bar{b}, r_{12}] = 1$ means that the conjugation $b * r_{12}$ equals r_{12}. This implies that the subsets $\{1, 2\}$ and $\{3, \dots, l-1\}$ are invariant under the action by $\bar{b}$. This means that $\bar{b}$ lies in the subgroup of $\mathsf{W}(\mathbf{A}_{l-2})$ generated by $a_1; a_3, \dots, a_{l-2}$. In other words, there is a word representing $\bar{b}$ which does not contain a_2. Moreover, the general theory of Coxeter groups (see [21, 7]) implies that there exists a W-*reduced word* w with this property (see *ibid*). Now, the primitivity of b implies that the same word w is a word representing the braid b. Now, since the generators $a_1; a_3, \dots, a_{l-2}$ commute with a_1 and a_0, we get the desired property $[b, a_1a_0^{-1}] = 1$.

Next, let us observe that every braid $b \in \mathsf{Br}(\mathbf{A}_{l-2})$ can be factorised as $b = b' \cdot b''$ where b' is primitive and b'' is a pure braid. Now we can conclude the following relation:

(R3) $[b, a_0a_1^{-1}] = 1$ provided $b * r_{12} = r_{12}$ for any braid $b \in \mathsf{Br}(\mathbf{A}_{l-2})$.

This relation can be read as follows:
(R3′) If $b_1 * r_{12} = b_2 * r_{12}$ for two braids b_1, b_2, then $b_1 * (a_0a_1^{-1}) = b_2 * (a_0a_1^{-1})$.

The next relation we shall need is:

(R4) Let b_1, b_2, b_3 be three braids such that $b_1 * r_{12} = r_{ij}$, $b_2 * r_{12} = r_{jk}$, and $b_3 * r_{12} = r_{ik}$, so that $r_{ij} + r_{jk} = r_{ik}$ in $\mathbb{Z}_2^{l-2}$. Then

$$\big(b_1 * (a_0a_1^{-1})\big) \cdot \big(b_2 * (a_0a_1^{-1})\big) = b_3 * (a_0a_1^{-1})$$

The proof of this property is done as follows. First, we observe that the symmetric group $\mathsf{W}(\mathbf{A}_{l-2})$ acts transitively on the triples (i, j, k). This implies that it is sufficient to prove this relation for the special case $(i, j, k) = (1, 2, 3)$. The latter reads

$$\big(a_0a_1^{-1}\big) \cdot \big((a_1a_2) * (a_0a_1^{-1})\big) = a_2 * (a_0a_1^{-1}).$$

Let us now calculating directly (again in the group generated by a_0, a_1, a_2) using the relations established above and denoting $a_1 =: a$, $a_2 =: b$, $a_0 =: c$ we get

$$\begin{aligned}(a_0 a_1^{-1}) \cdot \big((a_1 a_2) * (a_0 a_1^{-1})\big) &= ca^{-1}ab\,ca^{-1}b^{-1}a^{-1} = cbc\,a^{-1}b^{-1}a^{-1} = bcb\,b^{-1}a^{-1}b^{-1} \\ &= b * (ca^{-1}) = a_2 * (a_0 a_1^{-1})\end{aligned}$$

Finally, we need the following relation:

(R5) Let b_1, b_2 be two braids such that $b_1 * r_{12} = r_{ij}$ and $b_2 * r_{12} = r_{km}$ with pairwise distinct i, j, k, m. In particular, r_{ij} and r_{km} commute in $\mathbb{Z}_2^{l-2}$. Then $b_1 * (a_0 a_1^{-1})$ and $b_2 * (a_0 a_1^{-1})$ commute.

As above, conjugation allows us to assume that $(i, j, k, m) = (1, 2, 3, 4)$. Let us denote $\tilde{r}_{ij} := b * (a_0 a_1^{-1})$ in the case when $b * r_{12} = r_{ij}$. Notice that we have $\tilde{r}_{ij} = \tilde{r}_{ji}$ and $\tilde{r}_{ij}^2 = 1$. Now we using (R4) get

$$\tilde{r}_{12}\tilde{r}_{34} = \tilde{r}_{12}\tilde{r}_{34}\tilde{r}_{12}^2 = \underline{\tilde{r}_{12}(\tilde{r}_{23}}\ \underline{\tilde{r}_{24})\tilde{r}_{12}}\tilde{r}_{12} = \underline{\tilde{r}_{13}\tilde{r}_{14}}\tilde{r}_{12} = \tilde{r}_{34}\tilde{r}_{12}$$

where each underlined expression is changed using (R4).

Now we are ready to give a proof of the theorem. By above, there exists a surjective homomorphism $\mathsf{Br}(\mathbf{D}_{l-1}) \to \mathit{SMap}(X, \xi)$.

Let x be any element of $\mathsf{Br}(\mathbf{D}_{l-1})$ which is identical in $\mathit{SMap}(X, \xi)$. Then x equals 1 in $\mathsf{Br}(\mathbf{A}_{l-2})$. Thus by Lemma 5.2 it can be written as a product of $b_i * (a_0 a_1^{-1})$ and $b_j * (a_0 a_1^{-1})^{-1}$ for some braids b_i, b_j. Moreover, the evaluation of such a product in $\mathsf{W}(\mathbf{D}_{l-1})$ is 1. We must show that $x \equiv 1$ modulo the relations (R1) and (R2). However, the relation (R3') says that the value of each factor $b_i * (a_0 a_1^{-1})$ modulo (R1) and (R2) depends only on the value $b_i * r_{12}$ in the group $\mathbb{Z}_2^{l-2}$, whereas the relations (R2–5) ensure that the value of the factorisation depends only on the value in $\mathbb{Z}_2^{l-2}$. $\square$

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