ADDITIVE ACTIONS ON COMPLETE TORIC SURFACES AND ON UNIQUENESS OF ADDITIVE ACTIONS ON COMPLETE TORIC VARIETIES

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ABSTRACT. By an additive action on an algebraic variety X we mean a regular effective action $\mathbb{G}_a^n \times X \to X$ with an open orbit of the commutative unipotent group $\mathbb{G}_a^n$. In these two papers, we give a classification of additive actions on complete toric surfaces and a uniqueness criterion for additive action on a complete toric variety.

1. INTRODUCTION

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Denote its additive group by $\mathbb{G}_a = (\mathbb{K}, +)$. Consider the commutative unipotent group $\mathbb{G}_a^n = \mathbb{G}_a \times \ldots \times \mathbb{G}_a$ (n times). By an additive action on an irreducible algebraic variety X of dimension n we mean an effective regular action $\mathbb{G}_a^n \times X \to X$ with an open orbit. If a complete variety X admits an additive action, we can consider variety X as an equivariant completion of affine space $\mathbb{A}^n$ with respect to the group of parallel translations on $\mathbb{A}^n$.

A systematic study of additive actions began with the work of Hassett and Tschinkel [20]. They introduced a correspondence between additive actions on the projective space $\mathbb{P}^n$ and local $(n+1)$-dimensional commutative associative algebras with a unit; see also [21, Proposition 5.1] for a more general result. Hassett-Tschinkel correspondence allows to obtain the classification of additive actions on projective space $\mathbb{P}^n$ for $n \leq 5$; these are precisely the cases when the number of additive actions is finite.

The study of additive actions was originally motivated by Manin's conjecture about the distribution of rational points of bounded height on algebraic varieties, see works of Chambert-Loir and Tschinkel [8, 9].

There are some classification results for additive actions on various classes of varieties, in particular, on flag varieties [1, 14, 17, 18], singular del Pezzo surfaces [13], Hirzebruch surfaces [20], and weighted projective planes [2].

Some results in this direction are devoted to the uniqueness of additive actions. In [23], it is proved that an additive action on a smooth nondegenerate projective quadric is unique up to isomorhpism. Uniqueness of an additive action on a flag variety that is not isomorphic to a projective space is proved indepently and by completely different methods in [18] and [14].

The present work concerns the uniqueness of additive actions in the case of toric varieties. This problem was raised in [7, Section 6]. In [12], it is proved that $\mathbb{G}_a$-actions on a toric variety X normalized by the acting torus T are in bijection with some special elements in the character lattice of the torus T called Demazure roots of the corresponding fan Σ. Let $\mathcal{R}(X)$ be the Cox ring of the variety X. Cox [10] noted that normalized $\mathbb{G}_a$-actions on a toric variety can be interpreted as certain $\mathbb{G}_a$-subgroups of automorphisms of the ring $\mathcal{R}(X)$. In turn, such subgroups correspond to homogeneous locally nilpotent derivations of this ring.

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In [6], all toric varieties admitting an additive action are described in terms of their fans. It is proved that if a complete toric variety X admits an additive action, then it admits an additive action normalized by the acting torus. Moreover, any two normalized additive actions on X are isomorphic.

This work is made up on the basis of two papers [15] and [16]. In [15], all additive actions on a complete toric surface were classified. It turns out that there are no more than two non-isomorphic additive actions on a complete toric surface, see Section 6. In [16], a criterion of uniqueness on additive actions on toric varieties was proved, see Section 7.

After presenting some preliminaries on toric varieties and Cox ring (Section 2) and $\mathbb{G}_a$-actions and Demazure roots (Section 3), we describe the results of [6] (Section 4). In Section 5, we prove some facts on Demazure roots of a toric variety admitting an additive action. In Section 6, we classify additive actions on complete toric surfaces. In Section 7, we prove the criterion on uniqueness of additive actions on toric varieties. Finally, in Section 8 we give some corollaries and examples and discuss the case of toric surfaces.

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2. Toric varieties and Cox rings

In this section, we introduce basic notation of toric geometry, see [11, 19] for details.

Definition 1. A *toric variety* is a normal variety X containing a torus $T \simeq (\mathbb{K}^\times)^n$ as a Zariski open subset such that the action of T on itself extends to an action of T on X.

Let M be the character lattice of T and N be the lattice of one-parameter subgroups of T. Let $\langle \cdot\,, \cdot \rangle : N \times M \to \mathbb{Z}$ be the natural pairing between the lattice N and the lattice M. It extends to the pairing $\langle \cdot\,, \cdot \rangle_{\mathbb{Q}} : N_{\mathbb{Q}} \times M_{\mathbb{Q}} \to \mathbb{Q}$ between the vector spaces $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2. A *fan* Σ in the vector space $N_{\mathbb{Q}}$ is a finite collection of strongly convex polyhedral cones σ such that

(1) for all cones $\sigma \in \Sigma$, each face of σ is also in Σ;
(2) for all cones $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of the cones σ_1 and σ_2.

There is a correspondence between toric varieties X and fans Σ in the vector space $N_{\mathbb{Q}}$, see [11, Section 3.1] for details.

Here, we recall basic notions of the Cox construction, see [3, Chapter 1] for more details. Let X be a normal variety. Suppose that the variety X has a free finitely generated divisor class group $\mathrm{Cl}(X)$ and there are only constant invertible regular functions on X. Denote the group of Weil divisors on X by $\mathrm{WDiv}(X)$ and consider a subgroup $K \subseteq \mathrm{WDiv}(X)$ which maps onto $\mathrm{Cl}(X)$ isomorphically. The *Cox ring* of the variety X is defined as

$$R(X) = \bigoplus_{D \in K} H^0(X, D), \text{ where } H^0(X, D) = \{f \in \mathbb{K}(X)^\times \mid \mathrm{div}(f) + D \geqslant 0\} \cup \{0\}$$

and the multiplication on homogeneous components coincides with the multiplication in the field of rational functions $\mathbb{K}(X)$ and extends to the Cox ring $R(X)$ by linearity. It is easy to see that up to isomorphism the graded ring $R(X)$ does not depend on the choice of the subgroup K.

Suppose that the Cox ring $R(X)$ is finitely generated. Then $\overline{X} := \operatorname{Spec} R(X)$ is a normal affine variety with an action of the torus $H_X := \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$. There is an open H_X-invariant subset $\widehat{X} \subseteq \overline{X}$ such that the complement $\overline{X} \backslash \widehat{X}$ is of codimension at least two in $\overline{X}$, there exists a good quotient $\pi_X\colon \widehat{X} \to \widehat{X}/\!/H_X$, and the quotient space $\widehat{X}/\!/H_X$ is isomorphic to X, see [3, Construction 1.6.3.1]. Thus, we have the following diagram:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\ i\ } & \overline{X} = \operatorname{Spec} R(X) \\ \big\downarrow{\scriptstyle /\!/H_X} & & \\ X & & \end{array}$$

It is proved in [10] that if X is toric, then $R(X)$ is a polynomial algebra $\mathbb{K}[x_1,\ldots,x_m]$, where the variables x_i correspond to T-invariant prime divisors D_i on X or, equivalently, to the rays ρ_i of the corresponding fan Σ. The $\mathrm{Cl}(X)$-grading on $R(X)$ is given by $\deg(x_i) = [D_i]$. In this case, $\overline{X}$ is isomorphic to $\mathbb{K}^m$, and $\overline{X} \setminus \widehat{X}$ is a union of some coordinate subspaces in $\mathbb{K}^m$ of codimension at least two. Denote the torus $(\mathbb{K}^*)^m$ acting diagonally on the variety $\overline{X}$ by $\mathbb{T}$. Therefore, there are two gradings on $R(X)$, namely, $\mathbb{Z}^m$-grading which corresponds to the $\mathbb{T}$-action and $\mathrm{Cl}(X)$-grading which corresponds to H_X-action.

Let us desribe a connection between the gradings by the group $\mathrm{Cl}(X)$ and by the group $\mathbb{Z}^m$ on $R(X)$. Each $w \in M$ gives a character $\chi^w : T \to \mathbb{K}^*$, and hence, χ^w is a rational function on X. By [11, Theorem 4.1.3], the function χ^w defines a principal divisor $\mathrm{div}(\chi^w) = -\sum_\rho \langle p_\rho, w\rangle D_\rho$. Let us consider a map $M \longrightarrow \mathbb{Z}^m$ defined by $w \mapsto (\langle p_1, w\rangle, \ldots, \langle p_m, w\rangle)$, where $\rho_1,\ldots,\rho_m$ are one-dimensional cones of Σ and p_i are primitive vectors on rays ρ_i. By [19, §3.4], this map gives an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^m \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Here, a divisor $D \in \mathbb{Z}^{\Sigma(1)} = \mathbb{Z}^m$ determines an element $[D] \in \mathrm{Cl}(X)$. We identify the group $\mathbb{Z}^m$ with the character lattice of the torus $(\mathbb{K}^*)^m$. Thus, every element $w \in M$ corresponds to the character $\overline{\chi}^w$ of the torus $\mathbb{T}$. Moreover, for any $w, w' \in M$ the equality $w = w'$ holds if and only if $\overline{\chi}^w = \overline{\chi}^{w'}$.

3. Demazure roots and locally nilpotent derivations

Let X be a toric variety of dimension n, and Σ be the fan of the variety X. Let $\Sigma(1) = \{\rho_1,\ldots,\rho_m\}$ in N be the set of rays of the fan Σ and p_i be the primitive lattice vector on the ray ρ_i.

For any ray $\rho_i \in \Sigma(1)$, we consider the set $\mathfrak{R}_i$ of all vectors $e \in M$ such that

(1) $\langle p_i, e\rangle = -1$ and $\langle p_j, e\rangle \geq 0$ for $j \neq i$, $1 \leq j \leq n$;
(2) if σ is a cone of Σ and $\langle v, e\rangle = 0$ for all $v \in \sigma$, then the cone generated by σ and ρ_i is in Σ as well.

Elements of the set $\mathfrak{R} = \bigcup_{i=1}^{m} \mathfrak{R}_i$ are called *Demazure roots* of the fan Σ (see [12, Section 3.1] or [22, Section 3.4]). Let us divide the roots $\mathfrak{R}$ into two classes:

$$\mathfrak{S} = \mathfrak{R} \cap -\mathfrak{R}, \quad \mathfrak{U} = \mathfrak{R} \setminus \mathfrak{S}.$$

Roots in $\mathfrak{S}$ and $\mathfrak{U}$ are called *semisimple* and *unipotent*, respectively.

A derivation ∂ of an algebra A is said to be *locally nilpotent* if for every $f \in A$, there exists $k \in \mathbb{N}$ such that $\partial^k(f) = 0$. For any locally nilpotent derivation ∂ on A, the map $\varphi_\partial : \mathbb{G}_a \times A \to A$, $\varphi_\partial(s,f) = \exp(s\partial)(f)$ defines a structure of a rational $\mathbb{G}_a$-algebra on A. A derivation ∂ on a graded ring $A = \bigoplus_{\omega\in K} A_\omega$ is said to be *homogeneous* if it respects the K-grading. If $f, h \in A\setminus \ker\partial$ are homogeneous, then $\partial(fh) = f\partial(h) + \partial(f)h$ is homogeneous too, and $\deg \partial(f) - \deg f$ is equal to $\deg \partial(h) - \deg h$. Thus, any homogeneous derivation ∂ has a well-defined *degree* given as $\deg \partial = \deg \partial(f) - \deg f$ for any homogeneous element $f \in A \setminus \ker \partial$.

Every locally nilpotent derivation of $\mathrm{Cl}(X)$-degree zero on the Cox ring $R(X)$ induces a regular action $\mathbb{G}_a \times X \to X$. In fact, any regular $\mathbb{G}_a$-action on X arises this way, see [10, Section 4] and [3, Theorem 4.2.3.2]. If a $\mathbb{G}_a$-action on a variety X is normalized by the acting torus T, then the lifted $\mathbb{G}_a$-action on $\overline{X} = \mathbb{K}^m$ is normalized by the diagonal torus $\mathbb{T}$. Conversely, any $\mathbb{G}_a$-action on $\mathbb{K}^m$ normalized by the torus $\mathbb{T}$ and commuting with the subtorus H_X induces a $\mathbb{G}_a$-action on X. This shows that $\mathbb{G}_a$-actions on X normalized by the torus T are in bijection with locally nilpotent derivations of the Cox ring $\mathbb{K}[x_1, \ldots, x_m]$ that are homogeneous with respect to the grading by the lattice $\mathbb{Z}^m$ and have degree zero with respect to the $\mathrm{Cl}(X)$-grading.

For any element $e \in \mathfrak{R}_i$, we consider the locally nilpotent derivation $\partial_e = \prod_{j \neq i} x_j^{\langle p_j, e\rangle} \frac{\partial}{\partial x_i}$ on the algebra $R(X)$. This derivation has degree zero with respect to the grading by the group $\mathrm{Cl}(X)$. This way one obtains a bijection between Demazure roots in $\mathfrak{R}$ and locally nilpotent derivations on the ring $R(X)$ which are homogeneous with respect to $\mathbb{Z}^m$-grading and have degree zero with respect to the $\mathrm{Cl}(X)$-grading. The latter ones, in turn, are in bijection with $\mathbb{G}_a$-actions on X normalized by the acting torus.

Proposition 1. *[10, Proposition 4.4] There is a one-to-one correspondence*

$$\mathfrak{R}_i \leftrightarrow \{(x_i, x^D) : x^D \in R(X) \text{ is a monomial, } x^D \neq x_i,\ \deg(x^D) = \deg(x_i)\}.$$

Corollary 1. *If a homogeneous component C of the Cox ring $R(X)$ contains a variable x_i, then the vector space C is spanned by x_i and $\partial_e(x_i)$, where e runs over $\mathfrak{R}_i$.*

4. Complete toric varieties admitting an additive action

In this section, we shortly present the results of [6]. Let X be a toric variety of dimension n admitting an additive action, and Σ be the fan of the variety X.

Since the variety X admits an additive action, the variety X contains an open $\mathbb{G}_a^n$-orbit isomorphic to the affine space $\mathbb{K}^n$. By [4, Lemma 1], any invertible function on the variety X is constant and the divisor class group $\mathrm{Cl}(X)$ is freely generated. In particular, the Cox ring $R(X)$ introduced in Section 2 is well defined.

We denote primitive vectors on the rays of the fan Σ by p_i, where $1 \leq i \leq m$.

Definition 3. A set $e_1, \ldots, e_n$ of Demazure roots of a fan Σ of dimension n is called a *complete collection* if $\langle p_i, e_j\rangle = -\delta_{ij}$, where $1 \leq i, j \leq n$ for some ordering of $p_1, \ldots, p_m$.

An additive action on a toric variety X is said to be *normalized* if the image of the group $\mathbb{G}_a^n$ in $\mathrm{Aut}(X)$ is normalized by the acting torus T.

Theorem 1. *[6, Theorem 1] Let X be a toric variety. Then normalized additive actions on X are in bijection with complete collections of Demazure roots of the fan Σ.*

Corollary 2. *A toric variety X admits a normalized additive action if and only if there is a complete collection of Demazure roots of the fan Σ.*

Theorem 2. *[6, Theorem 2] Any two normalized additive actions on a toric variety are isomorphic.*

Theorem 3. *[6, Theorem 3] Let X be a complete toric variety. The following conditions are equivalent:*

(1) *there exists an additive action on* X*;*
(2) *there exists a normalized additive action on* X*;*
(3) *a maximal unipotent subgroup* U *of the automorphism group* $\mathrm{Aut}(X)$ *acts on* X *with an open orbit.*

Definition 4. The *negative octant* of the rational vector space V with respect to a basis $f_1, \dots, f_n$ is the cone $\left\{ \sum\limits_{i=1}^{n} \lambda_i f_i \mid \lambda_i \leq 0 \right\} \subset V$.

Proposition 2. *Let* X *be a complete toric variety. The following statements are equivalent:*
(1) *there exists an additive action on* X*;*
(2) *we can order rays of the fan* Σ *in such a way that the primitive vectors on the first* n *rays form a basis of the lattice* N*, and the remaining rays lie in the negative octant with respect to this basis.*

Proof. We prove (1) $\Rightarrow$ (2). Let us check that $e_1, \dots, e_n$ is a complete collection of Demazure roots, then $p_1, \dots, p_n$ form a basis of the lattice N. Assume that the vectors $p_1, \dots, p_n$ are linearly dependent, i.e. there exists a non-trivial linear relation $\alpha_1 p_1 + \ldots + \alpha_n p_n = 0$. Then we get $-\alpha_i = \langle \alpha_1 p_1 + \ldots + \alpha_n p_n, e_i \rangle = 0$ for all $1 \leq i \leq n$, a contradiction. Consider an arbitrary vector $v = \sum_{i=1}^{n} \nu_i p_i$ of the lattice N. By definition of a complete collection, we get $\langle v, e_i \rangle = -\nu_i \in \mathbb{Z}$. Therefore, the vectors $p_1, \dots, p_n$ form the basis of the lattice N.

All other vectors p_j, $j > n$, are equal to $-\sum_{l=1}^{n} \alpha_{jl} p_l$ for some integer α_{jl}. By definition of a Demazure root, we obtain

$$0 \leq \langle p_j, e_i \rangle = \sum \alpha_{jl} \delta_{li} = \alpha_{ji}.$$

The converse implication is straightforward.

Equivalence (1) $\Leftrightarrow$ (3) follows from Theorems 1 and 3. $\square$

We can order p_i in such a way that the first n vectors form a basis of the lattice N and the remaining vectors p_j $(n < j \leq m)$ are equal to $\sum_{i=1}^{n} -\alpha_{ji} p_i$ for some non-negative integers α_{ji}.

Corollary 3. *The elements* $\deg(x_j), n < j \leq m$ *form a basis of* $\mathrm{Cl}(X) \simeq \mathbb{Z}^{m-n}$ *and an element* $\deg(x_i), 1 \leq i \leq n$ *is equal to* $\sum\limits_{j=n+1}^{m} \alpha_{ji} \deg(x_j)$.

Proof. The matrix of the linear map $M \to \mathbb{Z}^m$ in the basis $p_1^*, \dots, p_n^*$ in M and in the standart basis of the lattice $\mathbb{Z}^m$ is equal to $\begin{pmatrix} I_n \\ -A \end{pmatrix}$, where I_n is the identity matrix of size n and $A = (\alpha_{ji})$, $n < j \leq m, 1 \leq i \leq n$. Therefore, the elements $\deg(x_j), n < j \leq m$ form a basis of $\mathrm{Cl}(X) \simeq \mathbb{Z}^{m-n}$ and the elements $\deg(x_i)$ are equal to $\sum\limits_{j=n+1}^{m} \alpha_{ji} \deg(x_j)$.

$\square$

5. Demazure roots of a variety admitting an additive action

Let X be a complete toric variety of dimension n admitting an additive action, and Σ be the fan of the variety X. Denote the primitive vectors on the rays ρ_i of the fan Σ by p_i, where $1 \leq i \leq m$.

From Proposition 2 it follows that we can order p_i in such a way that the first n vectors form a basis of the lattice N and the remaining vectors p_j $(n < j \leq m)$ are equal to $\sum_{i=1}^{n} -\alpha_{ji}p_i$ for some non-negative integers α_{ji}. Let us denote the dual basis of the basis $p_1, \ldots, p_n$ by $p_1^*, \ldots, p_n^*$.

Lemma 1. *Consider $1 \le i \le n$. The set $\mathfrak{R}_i$ is a subset of the set $-p_i^* + \sum\limits_{l=1, l\neq i}^{n} \mathbb{Z}_{\ge 0} p_j^*$ and the vector $-p_i^*$ is contained in $\mathfrak{R}_i$.*

Proof. Let $e = \sum\limits_{i=1}^{n} \varepsilon_i p_i^*$ be a Demazure root from $\mathfrak{R}_i$. By the definition, the Demazure roots from $\mathfrak{R}_i$ are defined by the following equations:

$$\begin{array}{rl} \varepsilon_i = -1 & \\ \varepsilon_l \ge 0, & l \le n,\, l \ne i \\ \alpha_{ji} - \sum\limits_{\substack{l=1 \\ l \neq i}}^{n} \varepsilon_l \alpha_{jl} \ge 0, & n < j \le m \end{array} \tag{1}$$

It is clear that all possible solutions lie in the set $-p_i^* + \sum\limits_{\substack{l=1 \\ l\neq i}}^{n} \mathbb{Z}_{\ge 0} p_l^*$, and the vector $-p_i^*$ satisfies them. □

Consider the set $\mathrm{Reg}(\mathfrak{S}) = \{u \in N : \langle u, e\rangle \neq 0 \text{ for all } e \in \mathfrak{S}\}$. Any element u from the set $\mathrm{Reg}(\mathfrak{S})$ divides the set of semisimple roots $\mathfrak{S}$ into two classes as follows:

$$\mathfrak{S}_u^+ = \{e \in \mathfrak{S} : \langle u, e\rangle > 0\}, \quad \mathfrak{S}_u^- = \{e \in \mathfrak{S} : \langle u, e\rangle < 0\}.$$

At this point, any element of ${\mathfrak{S}_u}^+$ is called *positive* and any element of $\mathfrak{S}_u^-$ is called *negative.*

Proposition 3. *Let X be a complete toric variety admitting an additive action, and $\mathfrak{R} = \bigcup\limits_{i=1}^{m} \mathfrak{R}_i$ be the set of its Demazure roots. Then*

(1) *any element $e \in \mathfrak{R}_j, j > n$, is equal to $p_{i'}^*$ for some $1 \le i' \le n$;*
(2) *all unipotent Demazure roots lie in the set $\bigcup\limits_{i=1}^{n} \mathfrak{R}_i$;*
(3) *there exists a vector $u \in \mathrm{Reg}(\mathfrak{S})$ such that $\mathfrak{S}_u^+ \subset \bigcup\limits_{i=1}^{n} \mathfrak{R}_i$.*

Proof. We start with the first statement. Consider a root $e = \sum\limits_{i=1}^{n} \varepsilon_i p_i^* \in \mathfrak{R}_j$, where $j > n$. By definition of Demazure roots, we have $-\langle p_j, e\rangle = \sum\limits_{i=1}^{n} \alpha_{ji}\varepsilon_i = 1$ and $\varepsilon_i \ge 0$ for all $1 \le i \le n$. Consider the set $I_j = \{i : \alpha_{ji} > 0\}$. Then there exists $s \in I_j$ such that $\varepsilon_s = 1$ and for all $l \in I_j \setminus \{s\}$ the equality $\varepsilon_l = 0$ holds. Since X is complete, there is no half-space with all vectors p_i inside. Hence, for all $l \in \{1, \ldots, n\} \setminus I_j$ there exists $r > n$ such that $\alpha_{rl} > 0$. Since $\langle p_r, e\rangle = -\sum\limits_{i=1}^{n} \alpha_{ri}\varepsilon_i \ge 0$, we have $\varepsilon_l = 0$. This implies $e = p_s^*$. The first statement is proved.

Let us prove the second statement. As above, consider the root $e = p_{i_j}^* \in \mathfrak{R}_j$, $j > n$. From the first statement of Proposition 3 and Lemma 1 it follows that the element $-e$ is a root and lies in $\mathfrak{R}_{i_j}$ for some i_j. This means that the root e is semisimple. Hence, all unipotent roots lie in the set $\bigcup_{i=1}^{n} \mathfrak{R}_i$.

To prove (3), we should find a vector u from the set $\mathrm{Reg}(\mathfrak{S})$ such that the set $\bigcup_{j=n+1}^{m} \mathfrak{R}_j$ contains only negative roots. Consider the vector $u_0 = -\sum\limits_{i=1}^{n} p_i$. For every root $e \in \bigcup_{j=n+1}^{m} \mathfrak{R}_j$, we get the inequality $\langle u_0, e\rangle = -1 < 0$. We can add a small rational vector $\Delta u = \frac{1}{Q}\Delta u' \in N_{\mathbb{Q}}$, where $\Delta u' \in N$ and Q is a positive integer such that the inequality $\langle u_0 + \Delta u, e\rangle_{\mathbb{Q}} < 0$ holds for all roots $e \in \bigcup_{i=n+1}^{m} \mathfrak{R}_i$. So, we have $Q(u_0 + \Delta u) \in \mathrm{Reg}(\mathfrak{S})$, and we obtain the required vector $u := Q(u_0 + \Delta u)$. □

Now we recall basic definitions from the theory of partially ordered sets.

Definition 5. Consider a set P and a binary relation $\leq$ on P. Then $\leq$ is a *preorder* if it is reflexive and transitive; i.e., for all a, b and c in P, we have:

(1) $a \leq a$ (reflexivity);
(2) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

Two elements a, b are *comparable* if $a \leq b$ or $b \leq a$. Otherwise, they are *incomparable*. If every pair of different elements is incomparable, then the preorder is called *trivial*.

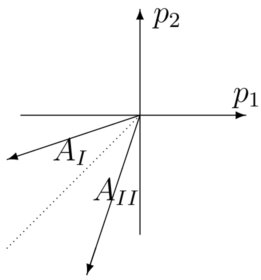
An element a in P is *maximal* if for any element b in P either $b \leq a$ or the elements a, b are incomparable.

Define a preorder $\leq$ on the set of rays $\{\rho_1, \ldots, \rho_n\}$ in the following way:

$$\rho_{i_1} \leq \rho_{i_2} \text{ if } \alpha_{ji_1} \leq \alpha_{ji_2} \text{ for every } n < j \leq m.$$

6. Additive actions on complete toric surfaces

Let X be a complete toric surface with the fan Σ. Suppose that X admits an additve action. Denote primitive vectors on the rays of the fan Σ by $p_1, \ldots, p_m$. By Proposition 2, we can assume that p_1, p_2 is the standard basis of $N_{\mathbb{Q}}$ and p_j, $j > 2$ is equal to $-\alpha_{j1}p_1 - \alpha_{j2}p_2$.



Definition 6. Let us call a fan Σ *wide* if it satisfies one of the following equivalent conditions:

(1) There exist $2 < j_1, j_2 \leq m$ such that $\alpha_{j_1 1} > \alpha_{j_1 2}$ and $\alpha_{j_2 1} < \alpha_{j_2 2}$;
(2) $\mathfrak{R}_1 = \{-p_1^*\}$ and $\mathfrak{R}_2 = \{-p_2^*\}$.

Proof of Equivalence. From the definition of Demazure roots it follows that

$$\mathfrak{R}_1 = \left\{(-1, k) : 0 \leq k \leq \min_{j>2}\left(\frac{\alpha_{j1}}{\alpha_{j2}}\right)\right\}, \quad \mathfrak{R}_2 = \left\{(k, -1) : 0 \leq k \leq \min_{j>2}\left(\frac{\alpha_{j2}}{\alpha_{j1}}\right)\right\}.$$

From this it follows that $|\mathfrak{R}_1| = \left\lfloor \min\limits_{j>2}\left(\frac{\alpha_{j1}}{\alpha_{j2}}\right)\right\rfloor + 1$, $|\mathfrak{R}_2| = \left\lfloor \min\limits_{j>2}\left(\frac{\alpha_{j2}}{\alpha_{j1}}\right)\right\rfloor + 1$. This implies the equivalence. □

Let us consider two areas in $N_{\mathbb{Q}}$:

$$A_I = \{(x, y) \in M_{\mathbb{Q}} : x \leq 0, y \leq 0, x < y\},$$
$$A_{II} = \{(x, y) \in M_{\mathbb{Q}} : x \leq 0, y \leq 0, x > y\}.$$

The first condition from the definition of a wide fan means that there is a ray of Σ in the area A_I and there is a ray in the area A_{II}.

Now we are ready to formulate the main theorem of this section.

Theorem 4. *Let X be a complete toric surface admitting an additive action. Then there is only one additive action on X if and only if the fan Σ is wide; otherwise there exist two non-isomorphic additive actions, one is normalized and the other is not.*

Proof of Theorem 4. We are going to classify additive actions on X by describing two-dimensional subgroups of a maximal unipotent subgroup U of the automorphism group $\mathrm{Aut}(X)$ up to conjugation in $\mathrm{Aut}(X)$.

Fix a vector $u \in \mathrm{Reg}(\mathfrak{S})$ that satisfies assertion (3) of Proposition 3. Hereafter, we write $\mathfrak{S}^+$ instead of $\mathfrak{S}^+_u$. Denote the set $\mathfrak{S}^+ \cup \mathfrak{U}$ by $\mathfrak{R}^+$. From Proposition 3 it follows that $\mathfrak{R}^+$ lies in the set $\bigcup_{i=1}^n \mathfrak{R}_i$. All the one-parameter subgroups of roots from $\mathfrak{R}^+$ generate the maximal unipotent subgroup U in the group $\mathrm{Aut}(X)$, see [10, Proposition 4.3]. Denote the set $\mathfrak{R}^+ \cap \mathfrak{R}_i$ by $\mathfrak{R}^+_i$.

Lemma 2. *There exists $i \in \{1,2\}$ such that $|\mathfrak{R}^+_i| = 1$. Moreover, $\max_{i=1,2} |\mathfrak{R}^+_i| = \max_{i=1,2} |\mathfrak{R}_i|$.*

Proof. From the definition of Demazure roots it follows that

$$\mathfrak{R}_1 = \left\{(-1,k) : 0 \le k \le \min_{j>2} \left(\frac{\alpha_{j1}}{\alpha_{j2}}\right)\right\}, \quad \mathfrak{R}_2 = \left\{(k,-1) : 0 \le k \le \min_{j>2} \left(\frac{\alpha_{j2}}{\alpha_{j1}}\right)\right\}.$$

We have $|\mathfrak{R}_1| > 1, |\mathfrak{R}_2| > 1$ simultaneously if and only if

$$\mathfrak{R}_1 = \{(-1,0),(-1,1)\}$$
$$\mathfrak{R}_2 = \{(0,-1),(1,-1)\}.$$

Since the roots $(-1,1),(1,-1)$ are opposite to each other, only one of them can lie in $\mathfrak{R}^+$.

Only the roots $(-1,1),(1,-1)$ can lie in the set $(\mathfrak{R}_1 \cap -\mathfrak{R}_2) \cup (\mathfrak{R}_2 \cap -\mathfrak{R}_1)$. Thus, we have $|\mathfrak{R}^+_1| = 1$, $\mathfrak{R}^+_2 = \mathfrak{R}_2$ or $|\mathfrak{R}^+_2| = 1$, $\mathfrak{R}^+_1 = \mathfrak{R}_1$. □

Without loss of generality, it can be assumed that $|\mathfrak{R}^+_1| = 1$. Denote the cardinality of the set $\mathfrak{R}^+_2$ by $d+1$. By Definition 6 the fan is wide if and only if d is equal to 0. In there term, we have $\mathfrak{R}^+_1 = \{(-1,0)\}$ and $\mathfrak{R}^+_2 = \{(k,-1) : 0 \le k \le d\}$. Denote LND that corresponds to the root $(-1,0) \in \mathfrak{R}^+_1$ by δ, and LNDs that correspond to roots $(k,-1) \in \mathfrak{R}^+_2, 0 \le k \le d$ by ∂_k.

Lemma 3. *The following equations hold:*

$$[\delta, \partial_k] = k\partial_{k-1}, \qquad [\partial_k, \partial_{k'}] = 0.$$

Proof. In this proof, we use notation introduced in Section 2. The correspondence between Demazure roots and LNDs implies:

$$\delta = \prod_{j=3}^m x_j^{\alpha_{j1}} \frac{\partial}{\partial x_1}, \quad \partial_k = x_1^k \prod_{j=3}^m x_j^{\alpha_{j2} - k\alpha_{j1}} \frac{\partial}{\partial x_2}$$

It can be easily checked that the derivations ∂_k commute with each other. Moreover, direct computations show that the commutator $[\delta, \partial_k]$ is equal to the derivation $k\partial_{k-1}$. □

Let us find all commutative subgroups in the group U that correspond to additive actions. Such groups are in bijection with some pairs (D_1, D_2) of commuting LNDs. Note that not every pair of commuting LNDs corresponds to an additive action.

Lemma 4. *In the above terms, there is an invertible linear operator ϕ on the vector space $\langle D_1, D_2\rangle$ that sends the derivations D_1, D_2 to*

$$\begin{cases} \phi(D_1) = \delta + \sum\limits_{k=0}^{d} \mu_k \partial_k \\ \phi(D_2) = \partial_0 \end{cases}, \qquad \mu_k \in \mathbb{K} \tag{2}$$

Proof. Every pair of derivations has the form $D_1 = \lambda^{(1)}\delta + \sum \mu_k^{(1)}\partial_k$ and $D_2 = \lambda^{(2)}\delta + \sum \mu_k^{(2)}\partial_k$. If $\lambda^{(1)} = \lambda^{(2)} = 0$, then dimension of the orbit in the total space X is less than 2. Therefore, the orbit can not become open after the factorization $\widehat{X} \to X$. Thus, without loss of generality we can assume that $\lambda^{(1)} \neq 0$. We can convert derivations D_1, D_2 to the form $\delta + \sum \mu_k^{(1)}\partial_k, \sum \mu_k^{(2)}\partial_k$. From Lemma 3 it follows that the derivations D_1, D_2 commute if and only if $\mu_k^{(2)} = 0$ for $k > 0$. Thus, we can convert derivations D_1, D_2 to the form $\delta + \sum \mu_k^{(1)}\partial_k, \mu_0^{(2)}\partial_0$, with $\mu_0^{(2)} \neq 0$. We can assume that $\mu_0^{(2)} = 1$. □

Lemma 5. *Every pair of derivations of form (2) corresponds to an additive action.*

Proof. Let us consider the $\mathbb{G}_a^2$-action corresponding to the LNDs D_1, D_2. We prove that the group $\mathbb{G}_a^2 \times H_X$ acts in the total space $\mathbb{K}^m$ with an open orbit. By construction, the group $\mathbb{G}_a^2$ changes exactly two of the coordinates $x_1, \dots, x_m$, while the weights of the remaining $m-2$ coordinates with respect to the $\mathrm{Cl}(X)$-grading form a basis of the lattice of characters of the torus H_X. From this it follows that there exists a point $p \in \mathbb{K}^m$ with trivial stabilizer. Due to $\dim(\mathbb{G}_a^2 \times H_X) = m$ we get that the orbit of the point p is open. □

Hereafter, we suppose that D_1, D_2 have form (2). From Lemma 4 it follows that if $d = 0$, then derivations D_1, D_2 can be converted to δ, ∂_0 respectively. Such LNDs correspond to a normalized additive action and every additive action is isomorphic to this action.

Hereafter, we assume that $d \neq 0$.

Lemma 6. *There exists an automorphism $\psi \in \mathrm{Aut}(R(X))$ that conjugates D_1, D_2 to the form*

$$\begin{cases} \psi(D_1) = \delta + \mu_d \partial_d \\ \psi(D_2) = \partial_0 \end{cases} \tag{3}$$

Proof. We are going to find numbers $\eta_k \in \mathbb{K}$ such that the automorphism $\psi = \exp(\delta + \sum\limits_{k=1}^{d} \eta_k\partial_k)$ is the desired one.

The automorphism ψ conjugates LNDs D_1, D_2 to the form

$$\exp(\delta + \sum_k \eta_k\partial_k) D_1 \exp(-\delta - \sum_k \eta_k\partial_k) =$$

$$= \mathrm{Ad}\left(\exp\left(\delta + \sum_k \eta_k\partial_k\right)\right) D_1 = \exp\left(\mathrm{ad}\left(\delta + \sum_k \eta_k\partial_k\right)\right) D_1 =$$

$$= D_1 + \sum_{l=1}^{\infty} \frac{\mathrm{ad}\left(\delta + \sum\limits_k \eta_k\partial_k\right)^l}{l!} D_1 = \delta + \sum_{k=0}^{d}\left(\mu_k + \sum_{l=1}^{d-k} \frac{(k+l)!}{k!}(-\mu_{k+l} + \eta_{k+l})\right)\partial_k;$$

$$\exp(\delta + \sum_k \eta_k\partial_k) D_2 \exp(-\delta - \sum_k \eta_k\partial_k) = D_2.$$

Here, we get the system of linear equations

$$\mu_k + \sum_{l=1}^{d-k} \frac{(k+l)!}{k!}(-\mu_{k+l} + \eta_{k+l}) = 0,\ 0 \le k \le d-1,$$

in variables $\eta_1, \dots, \eta_d$. This system has a unique solution as an upper triangular system and it is the solution we are looking for. $\square$

Hereafter, we suppose that D_1, D_2 have form (3). Thus, we have a family of additive actions parameterized by the number μ_d:

$$\begin{aligned} x_1 &\to \exp(s_1 D_1 + s_2 D_2) x_1 = x_1 + s_1 \prod_{j=3}^{m} x_j^{\alpha_{j1}} \\ x_2 &\to \exp(s_1 D_1 + s_2 D_2) x_2 = x_2 + (s_2 + \tfrac{\mu_d s_1^d}{d!}) \prod_{j=3}^{m} x_j^{\alpha_{j2}} + \sum_{k=1}^{d} \tfrac{\mu_d s_1^{d-k}}{k!} x_1^k \prod_{j=3}^{m} x_j^{\alpha_{j2} - k\alpha_{j1}} \end{aligned} \tag{4}$$

Note that every action corresponding to the pair of LNDs of form (3) acts on $x_j, 3 \le j \le m$ identically.

Lemma 7. *All additive actions with $\mu_d \neq 0$ are non-normalized and isomorphic to each other.*

Proof. We conjugate the pair of LNDs that have form (3) by an element t of the maximal torus $\mathbb{T} = (\mathbb{K}^*)^m$. One can easily see that see that for a homogeneous LND that corresponds to the Demazure root $e \in M$ and an element t of the maximal torus T we have $tD_e t^{-1} = \chi_e(t) D$. Indeed, the derivation D_e is equal to $\prod_{j \neq i} x_j^{\langle p_j, e \rangle} \frac{\partial}{\partial x_i}$, by definition. Let us consider the image $tD_e t^{-1}(x_i)$ of an element x_i. It is equal to $t_i^{-1} \prod_{j \neq i} t_j^{\langle p_j, e \rangle} \prod_{j \neq i} x_j^{\langle p_j, e \rangle}$. Thus, we get

$$tD_e t^{-1} = t_i^{-1} \prod_{j \neq i} t_j^{\langle p_j, e \rangle} D_e = \prod_{j=1}^{m} t_j^{\langle p_j, e \rangle} D_e = \overline{\chi}^e(t) D_e.$$

Using this fact we obtain

$$\begin{aligned} tD_1 t^{-1} &= \overline{\chi}^{(-1,0)}(t)\delta + \mu_d \overline{\chi}^{(d,-1)}(t)\partial_d \\ tD_2 t^{-1} &= \overline{\chi}^{(0,-1)}(t)\partial_0 \end{aligned}$$

Since $\overline{\chi}^{(-1,0)} \neq \overline{\chi}^{(d,-1)}$ we can conjugate an additive action with $\mu_d \neq 0$ to the additive action with $\mu_d = 1$. $\square$

From the last lemma it follows that there are two classes of additive actions. The first one ($\mu_d = 0$) is a normalized additive action:

$$\begin{aligned} x_1 &\to x_1 + s_1 \prod_{j=3}^{m} x_j^{\alpha_{j1}} \\ x_2 &\to x_2 + s_2 \prod_{j=3}^{m} x_j^{\alpha_{j2}}. \end{aligned} \tag{5}$$

The second is a non-normalized additive action:

$$\begin{aligned} x_1 &\to x_1 + s_1 \prod_{j=3}^{m} x_j^{\alpha_{j1}} \\ x_2 &\to x_2 + (s_2 + \tfrac{s_1^d}{d!}) \prod_{j=3}^{m} x_j^{\alpha_{j2}} + \sum_{k=1}^{d} \tfrac{s_1^{d-k}}{k!} x_1^k \prod_{j=3}^{m} x_j^{\alpha_{j2} - k\alpha_{j1}}. \end{aligned} \tag{6}$$

Lemma 8. *Actions* (5) *and* (6) *are not isomorphic.*

Proof. Let us consider the homogeneous component of $\mathbb{K}[\overline{X}]$ containing x_2:

$$C = \langle x_2\rangle \oplus \operatorname{span}\{x_1^k \prod_{j=3}^{m} x_j^{\alpha_{j2}-k\alpha_{j1}} : 0 \le k \le d\}.$$

We consider the space $V = \{s_1D_1 + s_2D_2 : s_1, s_2 \in \mathbb{K}\}$ and its subspace

$$\operatorname{Ann}_V f = \{v \in V : vf = 0\},\ f \in C.$$

Let $f = \lambda x_2 + \sum\limits_{k=0}^{d} \lambda_k x_1^k \prod\limits_{j=3}^{m} x_j^{\alpha_{j2}-k\alpha_{j1}}$ be an arbitrary non-zero element of C.

In the case of normalized action $(s_1D_1 + s_2D_2)f$ is equal to

$$s_2\lambda \prod_{j=3}^{m} x_j^{\alpha_{j2}} + s_1 \sum_{k=1}^{d} \lambda_k k x_1^{k-1} \prod_{j=3}^{m} x_j^{\alpha_{j2}-(k-1)\alpha_{j1}}.$$

Elements of $\operatorname{Ann}_V f$ are defined by the following equations:

$$\begin{aligned} \lambda s_2 + \lambda_1 s_1 &= 0 \\ \lambda_k s_1 &= 0, \qquad 2 \le k \le d \end{aligned} \tag{7}$$

The collection of subspaces $\operatorname{Ann}_V f$, where $f \in C \setminus \{0\}$, contains a family of lines $\{s_1D_1 + s_2D_2 : \lambda_1 s_1 + \lambda s_2 = 0\}, (\lambda : \lambda_1) \in \mathbb{P}^2$.

In the case of non-normalized action $(s_1D_1 + s_2D_2)f$ is equal to

$$s_2\lambda \prod_{j=3}^{m} x_j^{\alpha_{j2}} + s_1\lambda x_1^d \prod_{j=3}^{m} x_j^{\alpha_{j2}-d\alpha_{j1}} + s_1 \sum_{k=1}^{d} \lambda_k k x_1^{k-1} \prod_{j=3}^{m} x_j^{\alpha_{j2}-(k-1)\alpha_{j1}}.$$

Elements of $\operatorname{Ann}_V f$ are defined by the following equations:

$$\begin{aligned} \lambda s_2 + \lambda_1 s_1 &= 0 \\ \lambda_k s_1 &= 0, \qquad 2 \le k \le d \\ \lambda s_1 &= 0 \end{aligned} \tag{8}$$

The subspace $\operatorname{Ann}_V f$ for $f \in \mathbb{C} \setminus \{0\}$ can be either $\mathbb{K}D_2$ or 0.

Hence, actions (5) and (6) are not isomorphic. □

Remark 1. The idea of this proof is taken from the proof [2, Theorem 1].

In the case of a wide fan, Theorem 4 follows from Lemmas 4 and 5. In the case of a non-wide fan, we obtain the assertion from Lemmas 5-8. Theorem 4 is proved. □

7. On uniqueness of additive actions on complete toric varieties

Let X be a complete toric variety of dimension n admitting an additive action, and Σ be the fan of the variety X. Denote the primitive vectors on the rays ρ_i of the fan Σ by p_i, where $1 \le i \le m$. From Proposition 2 it follows that we can order p_i in such a way that the first n vectors form a basis of the lattice N and the remaining vectors p_j $(n < j \le m)$ are equal to $\sum_{i=1}^{n} -\alpha_{ji}p_i$ for some non-negative integers α_{ji}.

Fix a vector $u \in \operatorname{Reg}(\mathfrak{S})$ that satisfies assertion (3) of Proposition 3. Hereafter, we write $\mathfrak{S}^+$ instead of $\mathfrak{S}_u^+$. Denote the set $\mathfrak{S}^+ \cup \mathfrak{U}$ by $\mathfrak{R}^+$. From Proposition 3, it follows that the set $\mathfrak{R}^+$ lies in the set $\bigcup\limits_{i=1}^{n} \mathfrak{R}_i$. The one-parameter subgroups of roots from $\mathfrak{R}^+$

generate the maximal unipotent subgroup U in the group $\mathrm{Aut}(X)$ and $\dim U = |\mathfrak{R}^+|$, see [10, Proposition 4.3]. Denote the set $\mathfrak{R}^+ \cap \mathfrak{R}_i$ by $\mathfrak{R}_i^+$.

Let us denote a locally nilpotent derivation that corresponds to the Demazure root $e \in \mathfrak{R}$ by ∂_e.

Theorem 5. *Let X be a complete toric variety admitting an additive action. The following conditions are equivalent:*

(1) *the set $\mathfrak{R}_i$ is equal to $\{-p_i^*\}$ for every $1 \le i \le n$;*
(2) *the set $\mathfrak{R}^+$ is equal to $\{-p_1^*, \dots, -p_n^*\}$;*
(3) *the preorder $\le$ on the set of rays $\{\rho_1, \dots, \rho_n\}$ is trivial;*
(4) *any additive action on variety X is isomorphic to the normalized additive action.*

Proof. Equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 3.

Lemma 9. *The vector $-p_{i_1}^* + p_{i_2}^*$ is a Demazure root if and only if $\rho_{i_1} \ge \rho_{i_2}$.*

Proof. The element $-p_{i_1}^* + p_{i_2}^*$ is a Demazure root if and only if the element satisfies inequalities $\langle p_j, -p_{i_1}^* + p_{i_2}^* \rangle \ge 0$ for all $n < j \le m$ since $\langle p_i, -p_{i_1}^* + p_{i_2}^* \rangle \ge 0$ for $i \in \{1, \dots, n\} \setminus \{i_1\}$ and $\langle p_{i_1}, -p_{i_1}^* + p_{i_2}^* \rangle = -1$. The properties

$$\langle p_j, -p_{i_1}^* + p_{i_2}^* \rangle = \langle -\sum_{i=1}^n \alpha_{ji} p_i, -p_{i_1}^* + p_{i_2}^* \rangle = \alpha_{ji_1} - \alpha_{ji_2} \ge 0,$$

for $n < j \le m$ are equivalent to the properties $\alpha_{ji_1} \ge \alpha_{ji_2}$ for all $n < j \le m$, or to the property $\rho_{i_1} \ge \rho_{i_2}$. □

Let us prove implication (1) $\Rightarrow$ (3). Suppose the converse that $\rho_{i_1} \ge \rho_{i_2}$ for some $i_1 \ne i_2$. By Lemma 9, the vector $-p_{i_1}^* + p_{i_2}^*$ is a Demazure root and it lies in $\mathfrak{R}_{i_1}$, a contradiction.

Lemma 10. *Let e be a Demazure root from the set $\mathfrak{R}_i$ and $e \ne -p_i^*$. Then there exists a Demazure root $e' \in \mathfrak{R}_i$ with $e' = -p_i^* + p_r^*$ for some $1 \le r \le n$. Moreover, if $\langle p_r, e \rangle > 0$ for some r, then the vector $-p_i^* + p_r^*$ is a Demazure root.*

Proof. Let $e = -p_i^* + \sum\limits_{l=1, l\ne i}^n \varepsilon_l p_l^*$, where $\varepsilon_l = \langle p_l, e \rangle \ge 0$. There exists an index $r \ne i$ such that $\varepsilon_r \ne 0$. Let us define a vector $e' = -p_i^* + p_r^*$. We have $\langle p_j, e' \rangle \ge \langle p_j, e \rangle \ge 0$ for all $n < j \le m$. Thus, the element e' is a Demazure root. □

Let us prove implication (3) $\Rightarrow$ (1). Let us assume the converse. By Lemma 10, if the set $\mathfrak{R}_i$ is not equal to $\{-p_i^*\}$, then there exists r such that $-p_i^* + p_r^* \in \mathfrak{R}_i$. By Lemma 9, we get $\rho_i \ge \rho_r$, a contradiction.

Now we prove implication (2) $\Rightarrow$ (4). A maximal unipotent group U has dimension n. So, the subgroup U is the only candidate for $\mathbb{G}_a^n$ up to conjugation.

Let us prove implication (4) $\Rightarrow$ (3). Without loss of generality, let us assume that there exist rays ρ_1, ρ_2 such that $\rho_2 \le \rho_1$, where ρ_1 is a maximal ray. By Lemma 9, the vector $-p_1^* + p_2^*$ is a Demazure root. Let us consider the number $d = \max\{\varepsilon : -p_1^* + \varepsilon p_2^* \in \mathfrak{R}_1\}$ and take two ordered tuples of derivations:

$$D^{(1)} = (D_1^{(1)}, \dots, D_n^{(1)}) = (\partial_{-p_1^*}, \partial_{-p_2^*}, \partial_{-p_3^*}, \dots, \partial_{-p_n^*});$$
$$D^{(2)} = (D_1^{(2)}, \dots, D_n^{(2)}) = (\partial_{-p_1^*}, \partial_{-p_2^*} + \partial_{-p_1^* + dp_2^*}, \partial_{-p_3^*} \dots, \partial_{-p_n^*}).$$

Our goal is to show that these tuples correspond to non-isomorphic additive actions. To prove this fact, we find some invariant varieties $S_{V^{(q)}}(\mathcal{C}), q = 1, 2$, for the above mentioned additive actions and prove that these invariants are non-isomorphic. The variety $S_{V^{(q)}}(\mathcal{C}), q = 1, 2$, is a subset of Cox ring $R(X)$ connected with an additive action.

Firstly, we prove that the tuples $D^{(1)}$ and $D^{(2)}$ correspond to additive actions. The derivation $\partial_{-p_2^*} + \partial_{-p_1^*+dp_2^*}$ is a sum of two locally nilpotent derivations of degree zero with respect to $\mathrm{Cl}(X)$-grading. Therefore, any derivation in the tuples $D^{(q)}, q = 1, 2$, is a derivation of degree zero with respect to the $\mathrm{Cl}(X)$-grading.

Lemma 11. *Derivations in the tuples $D^{(1)}$ and $D^{(2)}$ are locally nilpotent.*

Proof. For any $1 \le i \le n$, the derivation $\partial_{-p_i^*}$ is locally nilpotent since it corresponds to a Demazure root. We should check that the derivation $\partial_{-p_2^*} + \partial_{-p_1^*+dp_2^*}$ is locally nilpotent. It easily follows from the following:

$$\begin{aligned}
&(\partial_{-p_2^*} + \partial_{-p_1^*+dp_2^*})(x_1) \in \mathbb{K}[x_2, x_{n+1}, x_{n+2}, \dots, x_m];\\
&(\partial_{-p_2^*} + \partial_{-p_1^*+dp_2^*})(x_2) \in \mathbb{K}[x_{n+1}, x_{n+2}, \dots, x_m];\\
&(\partial_{-p_2^*} + \partial_{-p_1^*+dp_2^*})(x_j) = 0, \text{ for } 2 < j \le m.
\end{aligned}$$

□

Lemma 12. *Derivations in the tuple $D^{(q)}, q = 1, 2$, pairwise commute.*

Proof. From Theorem 1 we know that derivations in the tuple corresponding to the normalized additive action commute, as a result $[\partial_{-p_i^*}, \partial_{-p_j^*}] = 0$. It remains to check that $[\partial_{-p_2^*} + \partial_{-p_1^*+dp_2^*}, \partial_{-p_i^*}] = [\partial_{-p_1^*+dp_2^*}, \partial_{-p_i^*}] = 0$ if $i \neq 2$. This can be checked directly. □

By these lemmas, we get that the ordered tuples $D^{(q)}, q = 1, 2$, correspond to actions $a^{(q)}$ on the variety X by the group $\mathbb{G}_a^n$.

Definition 7. Let us call an ordered tuple of locally nilpotent derivations $D = (D_1, \dots, D_n)$ *triangular* if $D_i x_i \neq 0$ and $D_l x_i = 0$ if $i > l$.

It is easy to check that the tuples of derivations $D^{(1)}$ and $D^{(2)}$ are triangular.

Lemma 13. *The $\mathbb{G}_a^n$-action corresponding to a triangular tuple of commuting locally nilpotent derivations has an open orbit on the variety X. Thus, a triangular ordered tuple of locally nilpotent derivations defines an additive action $\mathbb{G}_a^n \times X \to X$.*

Proof. We prove that there exists a point $p = (x_1, \dots, x_m) \in \widehat{X} \subset \overline{X}$ such that $\dim(\mathbb{G}_a \times H_X)p = m$. The Jacobian of the orbit morphism $\varphi_p \colon \mathbb{G}_a \times H_X \to \widehat{X}$ at the identity of the group $\mathbb{G}_a \times H_X$ is equal to $\prod\limits_{i=1}^{n} D_i x_i \prod\limits_{j=n+1}^{m} x_j$. There exists a point $p \in \widehat{X}$, where the product $\prod\limits_{i=1}^{n} D_i x_i \prod\limits_{j=n+1}^{m} x_j$ is not zero. The dimension of the tangent space of the orbit $(\mathbb{G}_a^n \times H_X)p$ at the point p is equal to $\dim \overline{X} = m$. Thus, the orbit $(\mathbb{G}_a^n \times H_X)p$ on the variety $\overline{X}$ is open. Consequently, after factorization $\pi_X : \widehat{X} \to X$ the orbit $\mathbb{G}_a^n \pi_X(p)$ is open on the variety X as well. □

Therefore, the action $a^{(q)}, q = 1, 2$, is an additive action.

Now we prove that actions corresponding to the tuples $D^{(1)}$ and $D^{(2)}$ are non-isomorphic. Let us consider an equivalence relation on the set of rays $\Sigma(1)$ determined by

$$\rho_{i_1} \sim \rho_{i_2} \iff \deg(x_{i_1}) = \deg(x_{i_2}) \text{ in } \mathrm{Cl}(X).$$

This partitions $\Sigma(1)$ into disjoint subsets $\bigsqcup\limits_{i=1}^{r}\Sigma(1)_i$, where each subset $\Sigma(1)_i$ corresponds to a set of variables of the same degree ω_i. Let $\mathcal{C}_i = \{f \in \mathcal{R}(X) : \deg(f) = \omega_i\}$ be the homogeneous component. Let us consider the vector space $\mathcal{C}_i$ as an algebraic variety $\mathbb{A}^{\dim \mathcal{C}_i}$. We take the algebraic variety $\mathcal{C} = \bigcup\limits_{i=1}^{r}\mathcal{C}_i$. We consider two vector spaces $V^{(1)} = \{\sum\limits_{i=1}^{n} s_i D_i^{(1)} : s_i \in \mathbb{K}\}$ and $V^{(2)} = \{\sum\limits_{i=1}^{n} s_i D_i^{(2)} : s_i \in \mathbb{K}\}$. For every element $f \in \mathcal{C}$, we regard the subspace $\operatorname{Ann}_V f = \{v \in V : vf = 0\}$ of a space V of derivations.

Let us consider the following sets:

$$S_V(\mathcal{C}_i) = \{f \in \mathcal{C}_i : \dim \operatorname{Ann}_V f \geq \dim V - 1\},$$

$$S_V(\mathcal{C}) = \{f \in \mathcal{C} : \dim \operatorname{Ann}_V f \geq \dim V - 1\} = \bigcup_{i=1}^{r} S_V(\mathcal{C}_i)$$

Lemma 14. *The subset $S_V(\mathcal{C}_i)$ is a closed subvariety of the variety $\mathcal{C}_i$.*

Proof. We have the system of linear equations $vf = 0$, where $v \in V$ and $f \in \mathcal{C}_i$ is a certain fixed element. We choose some bases in V and in $\mathcal{C}_i$. In these terms, the condition $\dim \operatorname{Ann}_f V \geq \dim V - 1$ means that the matrix of system of linear equations $vf = 0$ has rank less than 2 or, equivalently, every 2×2 submatrix is singular. Thus, $S_V(\mathcal{C}_i)$ is the subvariety of $\mathcal{C}_i$ defined by equations $\det(M) = 0$, where M runs over all 2×2 submatrices of the matrix of linear equation. □

By [11, Theorem 3.2.6] T-invariant divisors $D_1, \ldots, D_m$ on the variery X as well as the elements $[D_1], \ldots, [D_m] \in \operatorname{Cl}(X)$ are canonical. Therefore, the degrees of the variables are canonical, since the degrees are equal to $[D_1], \ldots, [D_m]$. As a result if additive actions $a^{(1)}, a^{(2)}$ are isomorphic, then the varieties $S_{V^{(1)}}(\mathcal{C}), S_{V^{(2)}}(\mathcal{C})$ should be isomorphic. We are going to prove that varieties $S_{V^{(1)}}(\mathcal{C})$ and $S_{V^{(2)}}(\mathcal{C})$ are not isomorphic.

Without loss of generality, we suppose $x_1 \in \mathcal{C}_1$.

Lemma 15. *For $i \neq 1$, we have $S_{V^{(1)}}(\mathcal{C}_i) = S_{V^{(2)}}(\mathcal{C}_i)$.*

Proof. We prove that the derivation $\partial_{-p_1^*+dp_2^*}$ is zero on the vector space $\mathcal{C}_i, i > 1$. Assume the converse. We know that $\partial_{-p_1^*+dp_2^*} = f\frac{\partial}{\partial x_1}$, where $f \in R(X)$. It follows that the derivation $\frac{\partial}{\partial x_1}$ is not zero on the vector space $\mathcal{C}_i$. There exists a certain variable $x_l \in \mathcal{C}_i$, $l \neq 1$. By Corollary 1, we get $\mathcal{C}_i = \{\lambda x_l + \sum_{e \in \mathfrak{R}_l} \lambda_e \partial_e(x_l) : \lambda, \lambda_e \in \mathbb{K}\}$. Since $l \neq 1$ we obtain $\frac{\partial}{\partial x_1}(x_l) = 0$. Also, from the definition of Demazure root we get $\partial_e(x_l) = x_1^{\langle p_1, e\rangle} g$, $g \in \mathbb{K}[x_2, \ldots, x_m]$. Since the ray ρ_1 is maximal, by Lemma 9 no vector $-p_l^* + p_1^*$ is a Demazure root. Then by Lemma 10 the pairing $\langle p_1, e\rangle$ is equal to zero and $\frac{\partial}{\partial x_1}(\partial_e(x_l)) = 0$, a contradiction.

As the derivation $\partial_{-p_1^*+dp_2^*}$ is zero, the tuples of derivations $D^{(1)}$ and $D^{(2)}$ are equal. □

By Corollary 1, for every element $f \in \mathcal{C}_1$, we can consider a representation $f = \lambda x_1 + \sum\limits_{e \in \mathfrak{R}_1} \lambda_e \partial_e(x_1)$ in the basis $x_1, \partial_e(x_1)$, where $e \in \mathfrak{R}_1$.

Since $\partial_{-p_i^*} = \prod_{l=n+1}^{m} x_l^{\alpha_{il}} \frac{\partial}{\partial x_i}$, $1 \leq i \leq n$, and $\partial_{-p_1^*+dp_2^*} = x_2^d \prod_{l=n+1}^{m} x_l^{\alpha_{1l} - d\alpha_{2l}} \frac{\partial}{\partial x_1}$, the image $D_i^{(q)}(\lambda x_1 + \sum_{e \in \mathfrak{R}_1} \lambda_e \partial_e(x_1)), 1 \leq i \leq n$ and $q = 1, 2$, belongs to $\operatorname{span}_{e \in \mathfrak{R}_1} \langle \partial_e(x_1)\rangle$. Let us introduce the coefficients $\upsilon_{e,i}^{(q)}$:

$$D_i^{(q)}\left(\lambda x_1 + \sum_{e\in\mathfrak{R}_1} \lambda_e \partial_e(x_1)\right) = \sum_{e\in\mathfrak{R}_1} \upsilon_{e,i}^{(q)} \partial_e(x_1).$$

Lemma 16. *The algebraic variety $S_{V^{(2)}}(\mathcal{C}_1)$ is the proper closed subset of the variety $S_{V^{(1)}}(\mathcal{C}_1)$.*

Proof. We prove that $S_{V^{(2)}}(\mathcal{C}_1) \subset \{\lambda = 0\}$. For this, we choose 2×2 submatrix $L = \begin{pmatrix} \upsilon^{(2)}_{-p_1^*,1} & \upsilon^{(2)}_{-p_1^*,2} \\ \upsilon^{(2)}_{-p_1^*+dp_2^*,1} & \upsilon^{(2)}_{-p_1^*+dp_2^*,2} \end{pmatrix}$. We have

$$\partial_{-p_1^*}(\lambda x_1 + \sum_{e\in\mathfrak{R}_1} \lambda_e \partial_e(x_1)) = \lambda \partial_{-p_1^*}(x_1),$$

$$(\partial_{-p_2^*} + \partial_{-p_1^*+dp_2^*})\left(\lambda x_1 + \sum_{e\in\mathfrak{R}_1} \lambda_e \partial_e(x_1)\right) = \lambda \partial_{-p_1^*+dp_2^*}(x_1) + \sum_{\substack{e\in\mathfrak{R}_1 \\ e+p_2^*\in\mathfrak{R}_1}} (\langle p_2, e\rangle + 1)\, \lambda_{e+p_2^*} \partial_e(x_1).$$

Since d is maximal with $-p_2^* + dp_1^*$ being a Demazure root, we have $\upsilon^{(2)}_{-p_1^*+dp_2^*,2} = \lambda$. The submatrix L is equal to $\begin{pmatrix} \lambda & \lambda_{-p_1^*+p_2^*} \\ 0 & \lambda \end{pmatrix}$. Thus, $S_{V^{(2)}}(\mathcal{C}_1) \subset \{\lambda = 0\}$. We know that $\operatorname{span}_{e\in\mathfrak{R}_1} \partial_e(x_1) \subset \ker \partial_{-p_2^*+dp_1^*}$. Therefore, if $\lambda = 0$ then the systems of linear equations are the same for tuples $D^{(1)}$ and $D^{(2)}$. This follows that

$$S_{V^{(2)}}(\mathcal{C}_1) = S_{V^{(2)}}(\mathcal{C}_1) \cap \{\lambda = 0\} = S_{V^{(1)}}(\mathcal{C}_1) \cap \{\lambda = 0\}.$$

Let us prove that $S_{V^{(1)}}(\mathcal{C}_1) \not\subset \{\lambda = 0\}$. Since $\sum s_i D_i^{(1)}(x_1) = s_1 \partial_{-p_1^*}(x_1)$ the point $\lambda = 1$ and all $\lambda_e = 0$ belongs to the variety $S_{V^{(1)}}(\mathcal{C}_1)$. $\square$

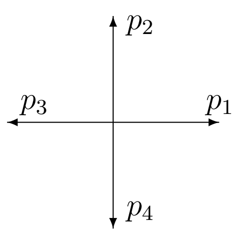
By Lemma 16, the varieties $S_{V^{(1)}}(\mathcal{C})$ and $S_{V^{(2)}}(\mathcal{C})$ are not isomorphic. This completes the proof of impication (1) $\Rightarrow$ (3). So, Theorem 5 is proved.

$\square$

8. Corollaries and examples

In this section, we preserve notation of the previous section. Here, we describe some examples illustrating Theorem 4.

Example 1. Let us consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Its fan is wide, and there is only one additive action up to isomorphism.

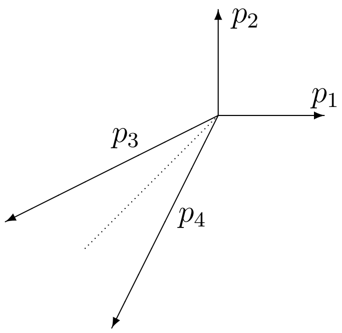


$$\begin{aligned} \mathfrak{R}_1 &= \{(-1,0)\} \\ \mathfrak{R}_2 &= \{(0,-1)\} \\ \mathfrak{R}_3 &= \{(1,0)\} \\ \mathfrak{R}_4 &= \{(0,1)\} \\ \mathfrak{R}^+ &= \{(-1,0),(0,-1)\} \end{aligned}$$

Normalized action:

$$\begin{aligned} x_1 &\to x_1 + s_1 x_3 \\ x_2 &\to x_2 + s_2 x_4 \\ x_3 &\to x_3 \\ x_4 &\to x_4 \\ (s_1, s_2) &\in \mathbb{G}_a^2 \end{aligned}$$

Example 2. Let us consider the surface corresponding to the following fan with $p_3 = -p_1 - 2p_2, p_4 = -2p_1 - p_2$. Its fan is wide, and there is only one additive action up to isomorphism.

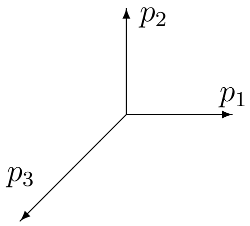


$$\begin{gathered}\mathfrak{R}_1 = \{(-1,0)\}\\ \mathfrak{R}_2 = \{(0,-1)\}\\ \mathfrak{R}_3 = \varnothing\\ \mathfrak{R}_4 = \varnothing\\ \mathfrak{R}^+ = \{(-1,0),(0,-1)\}\end{gathered}$$

Normalized action:

$$\begin{aligned} x_1 &\to x_1 + s_1x_3x_4^2\\ x_2 &\to x_2 + s_2x_3^2x_4\\ x_3 &\to x_3\\ x_4 &\to x_4\\ &(s_1,s_2)\in\mathbb{G}_a^2\end{aligned}$$

Example 3. Let us consider the projective plane $\mathbb{P}^2$. It corresponds to the following fan with $p_3 = -p_1 - p_2$. This fan is not wide. Therefore, there are two additive actions up to isomorphism.



$$\begin{gathered}\mathfrak{R}_1 = \{(-1,0),(-1,1)\}\\ \mathfrak{R}_2 = \{(0,-1),(1,-1)\}\\ \mathfrak{R}_3 = \{(1,0),(0,1)\}\\ \mathfrak{R}^+ = \{(-1,0),(0,-1),(1,-1)\}\end{gathered}$$

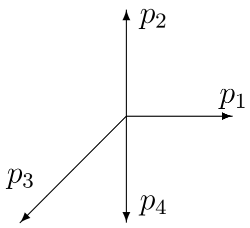
Normalized action:

$$\begin{aligned} x_1 &\to x_1 + s_1x_3\\ x_2 &\to x_2 + s_2x_3\\ x_3 &\to x_3\\ &(s_1,s_2)\in\mathbb{G}_a^2\end{aligned}$$

Non-normalized action:

$$\begin{aligned} x_1 &\to x_1 + s_1x_3\\ x_2 &\to x_2 + \tfrac{2s_2+s_1^2}{2}x_3 + s_1x_1\\ x_3 &\to x_3\\ &(s_1,s_2)\in\mathbb{G}_a^2\end{aligned}$$

Example 4. Let us consider Hirzebruch surface $\mathbb{F}_1$. It corresponds to the following fan with $p_3 = -p_1 - p_2, p_4 = -p_2$. This fan is not wide. Therefore, there are two additive actions up to isomorphism.



$$\begin{gathered}\mathfrak{R}_1 = \{(-1,0)\}\\ \mathfrak{R}_2 = \{(0,-1),(1,-1)\}\\ \mathfrak{R}_3 = \{(1,0)\}\\ \mathfrak{R}_4 = \varnothing\\ \mathfrak{R}^+ = \{(-1,0),(0,-1),(1,-1)\}\end{gathered}$$

Normalized action:

$$\begin{aligned} x_1 &\to x_1 + s_1x_3\\ x_2 &\to x_2 + s_2x_3x_4\\ x_3 &\to x_3\\ x_4 &\to x_4\\ &(s_1,s_2)\in\mathbb{G}_a^2\end{aligned}$$

Non-normalized action:

$$\begin{aligned} x_1 &\to x_1 + s_1x_3\\ x_2 &\to x_2 + \tfrac{2s_2+s_1^2}{2}x_3x_4 + s_1x_1x_4\\ x_3 &\to x_3\\ x_4 &\to x_4\\ &(s_1,s_2)\in\mathbb{G}_a^2\end{aligned}$$

For a geometric realization of these two actions, see [20, Propostion 5.5].

The next corollary follows from Theorem 5.

Corollary 4. *Let X be a complete toric variety admitting an additive action. The following conditions are equivalent:*

(1) *the dimension of a maximal unipotent subgroup of the automorphism group* $\mathrm{Aut}(X)$ *is equal to the dimension of the variety* X*;*
(2) *any additive action on* X *is isomorphic to the normalized additive action.*

Proof. The dimension of a maximal unipotent subgroup is equal to the size of the set $\mathfrak{R}^+$. □

Example 5. Let us consider the set of vectors $p_1, \ldots, p_5$ in $N = \mathbb{Z}^2$ such that the vectors p_1, p_2 form a basis of N, $p_3 = -p_1 + p_2$, $p_4 = -2p_1 - p_2$ and $p_5 = -p_1 - p_2$. Let $\rho_1, \ldots, \rho_5 \subset N_{\mathbb{Q}}$ be the rays generated by the vectors $p_1, \ldots, p_5$, respectively. Let us consider a complete toric variety X with the fan Σ such that $\Sigma(1) = \{\rho_1, \ldots, \rho_5\}$. It can be computed directly that $\mathfrak{R}_1 = \{-p_1^*, -p_1^* + p_2^*\}$ and $\mathfrak{R}_i = \varnothing, i \geq 2$. Therefore, a maximal unipotent subgroup of the group $\mathrm{Aut}(X)$ has dimension 2, but there is no additive action on the variety X by Lemma 1.

Now let us explain the connection between Theorem 4 and Theorem 5.

Corollary 5. *Let* X *be a complete toric variety admitting an additive action. The following conditions are equivalent:*
(1) *any additive action is isomorphic to the normalized additive action;*
(2) *the image under the projection along the coordinate plane* $\mathrm{span}\{p_1, \ldots, \widehat{p_{l_1}}, \ldots, \widehat{p_{l_2}}, \ldots, p_n\}$ *of the system of rays* $\Sigma(1)$ *to the plane spanned vectors* p_{l_1}, p_{l_2} *determines a wide fan for every* $1 \leq l_1 \neq l_2 \leq n$.

Proof. The image of the projection of the fan to the plane spanned by vectors p_{l_1}, p_{l_2} is wide if and only if the rays ρ_{l_1} and ρ_{l_2} are incomparable. Thus, the corollary stems from equivalence (3) $\Leftrightarrow$ (4) of Theorem 5. □

Corollary 6. *Let* X *be a complete toric variety admitting an additive action. If we have* $m = n+1$ *or, equivalently,* $\mathrm{rank}\,\mathrm{Cl}(X) = 1$*, then there are at least two non-isomorphic additive actions.*

Proof. By definition, the preorder on the rays $\rho_1, \ldots, \rho_n$ is the same as the natural order on numbers $\alpha_{n+1,1}, \ldots, \alpha_{n+1,n}$. Every two elements are comparable. Therefore, the preorder is not trivial. □

Corollary 6 covers the case of weighted projective spaces. By [6, Proposition 2], a weighted projective space $\mathbb{P}(a_0, \ldots, a_n), a_0 \leq a_1 \leq \ldots \leq a_n$ admits an additive action if and only if $a_0 = 1$. By this corollary, on a weighted projective space $\mathbb{P}(1, a_1, \ldots, a_n)$ there are at least two non-isomorphic additive actions.

The final example shows that in the case $m = n + 2$ an additive action can be unique.

Example 6. Let us consider the set of vectors $p_1, \ldots, p_{n+2}$ in $N = \mathbb{Z}^n$ such that the vectors $p_1, \ldots, p_n$ form a basis of N, $p_{n+1} = -\sum_{i=1}^n ip_i$ and $p_{n+2} = -\sum_{i=1}^n (n-i+1)p_i$. Let us consider the rays $\rho_1, \ldots, \rho_{n+2} \subset N_{\mathbb{Q}}$ generated by $p_1, \ldots, p_{n+2}$. We consider a complete toric variety X with a fan Σ such that $\Sigma(1) = \{\rho_1, \ldots, \rho_{n+2}\}$. By Theorem 5 an additive action on such a variety is unique.

References

[1] Ivan Arzhantsev. Flag varieties as equivariant compactifications of $\mathbb{G}_a^n$. Proc. Amer. Math. Soc. 139 (2011), no. 3, 783–786

[2] Ivan Arzhantsev, Sergey Bragin, and Yulia Zaitseva. Commutative algebraic monoid structures on affine spaces. Comm. Contem. Math., to appear; arXiv:1809.052911 [math.AG]
[3] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface. *Cox rings*. Cambridge Studies in Adv. Math. 144, Cambridge University Press, New York, 2015
[4] Ivan Arzhantsev, Alexander Perepechko, and Hendrik Süss. Infinite transitivity on universal torsors. J. London Math. Soc. 89 (2014), no. 3, 762–778
[5] Ivan Arzhantsev and Andrey Popovskiy. Additive actions on projective hypersurfaces. In: Automorphisms in Birational and Affine Geometry, Proc. Math. Stat. 79, Springer, 2014, 17-33
[6] Ivan Arzhantsev and Elena Romaskevich. Additive actions on toric varieties. Proc. Amer. Math. Soc. 145 (2017), no. 5, 1865–1879
[7] Ivan Arzhantsev and Elena Sharoyko. Hassett-Tschinkel correspondence: Modality and projective hypersurfaces. J. Algebra 348 (2011), no. 1, 217–232
[8] Antoine Chambert-Loir and Yuri Tschinkel. On the distribution of points of bounded height on equivariant compactifications of vector groups. Invent. Math. 148 (2002), no. 2, 421-452
[9] Antoine Chambert-Loir and Yuri Tschinkel. Integral points of bounded height on partial equivariant compactifications of vector groups. Duke Math. J. 161 (2012), no. 15, 2799–2836
[10] David Cox. The homogeneous coordinate ring of a toric variety. J. Alg. Geom. 4 (1995), no. 1, 17–50
[11] David Cox, John Little, and Henry Schenck. *Toric Varieties*. Graduate Studies in Math. 124, AMS, Providence, RI, 2011
[12] Michel Demazure. Sous-groupes algebriques de rang maximum du groupe de Cremona. Ann. Sci. Ecole Norm. Sup. 3 (1970), 507–588
[13] Ulrich Derenthal and Daniel Loughran. Singular del Pezzo surfaces that are equivariant compactifications. J. Math. Sciences 171 (2010), no. 6, 714–724
[14] Rostislav Devyatov. Unipotent commutative group actions on flag varieties and nilpotent multiplications. Transform. Groups 20 (2015), no. 1, 21–64
[15] Sergey Dzhunusov. Additive actions on toric surfaces, arXiv:1908.03563 [math.AG]
[16] Sergey Dzhunusov. On uniqueness of additive actions on complete toric varieties, arXiv:2007.10113 [math.AG]
[17] Evgeny Feigin. $\mathbb{G}_a^M$ degeneration of flag varieties. Selecta Math. New Ser. 18 (2012), no. 3, 513–537
[18] Baohua Fu and Jun-Muk Hwang. Uniqueness of equivariant compactifications of $\mathbb{C}^n$ by a Fano manifold of Picard number 1. Math. Res. Letters 21 (2014), no. 1, 121–125
[19] William Fulton. *Introduction to toric varieties*. Annales of Math. Studies 131, Princeton University Press, Princeton, NJ, 1993
[20] Brendan Hassett and Yuri Tschinkel. Geometry of equivariant compactifications of $\mathbb{G}_a^n$. Int. Math. Res. Notices 1999 (1999), no. 22, 1211–1230
[21] Friedrich Knop and Herbert Lange. Commutative algebraic groups and intersections of quadrics. Math. Ann. 267 (1984), no. 4, 555-571
[22] Tadao Oda. *Convex bodies and algebraic geometry: an introduction to toric varieties*. A Series of Modern Surveys in Math. 15, Springer Verlag, Berlin, 1988
[23] Elena Sharoiko. Hassett-Tschinkel correspondence and automorphisms of a quadric. Sbornik Math. 200 (2009), no. 11, 1715–1729

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