On the Index of Bipolar Surfaces to Otsuki Tori

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Abstract

We obtain estimates on the Morse index and nullity of bipolar surfaces to Otsuki tori $\tilde{O}_{p/q}$ for p/q sufficiently close to $\sqrt{2}/2$.

1 Introduction

A minimal surface is a critical point of the area functional and the index of a minimal surface is the Morse index of the corresponding critical point. Despite the naturalness and importance of this notion, the index is known only for a few examples of minimal surfaces. Unfortunately, there is no universal method for computing the index. In fact, for each case one should invent a new approach.

In the present paper we consider minimal surfaces in $\mathbb{S}^4$ called *bipolar surfaces to Otsuki tori.* Otsuki tori are minimal surfaces in $\mathbb{S}^3$. They first appeared in [Ôts70] but the original definition was implicit and very complicated. Later the definition was simplified in works [HLJ71] and [Pen13]. The second work contains also the study of extremal spectral properties of Otsuki tori.

Bipolar surfaces to Otsuki tori are obtained by applying the construction from [LJ70, §11] to Otsuki tori. Generally, this construction gives a minimal surface in $\mathbb{S}^5$ but in the case of Otsuki tori this surface always lies in some equatorial sphere $\mathbb{S}^4 \subset \mathbb{S}^5$. The study of extremal spectral properties of bipolar surfaces to Otsuki tori was done in [Kar14]. Moreover, this work contains a convenient parametrization of bipolar surfaces to Otsuki tori (see §2.2 and also Remark 1).

Our main result is Theorem 1 where we obtain estimates on the index and nullity of a bipolar surface to Otsuki tori under the condition that this surface is "close enough to the Clifford torus" (these words are clarified in §2.2). Unfortunately, our method allows neither compute the index exactly nor at least give an example of surface for which our estimates holds. However we hope that the method provides some intuition and can be easily adopted for numerical computations. For example, we are able to compute numerically the index of $\tilde{O}_{2/3}$, which is one of the surfaces of the family. We suppose that finding the exact value of index is a very difficult problem. See Remark 3 for the discussion of our main result.

The plan of the proof of Theorem 1 is the following. First we compute the Jacobi stability operator in appropriate local coordinates. Then, using the separation of variables, we reduce the spectral problem for Jacobi operator to the spectral problem for some periodic matrix Sturm-Liouville problem. Finally, we attack this problem using methods of the classical work [Edw64]. Thus our approach is a direct generalization of one used in [Pen12, Pen13, Kar14]. Note however that in our case the situation is complicated by the fact that bipolar surfaces to Otsuki tori are of codimension 2 in $\mathbb{S}^4$. This leads to multidimensional (matrix) Sturm-Liouville problem, which is much more complicated than the one-dimensional (scalar) one appeared in the works cited above. For an alternative approach to the codimension 2 case based on complex geometry see [KW18, Med21, Med22].

2 Preliminaries

2.1 Index and nullity of a minimal submanifold

In this section we fix our notation and recall the definitions of index and nullity of a minimal submanifold.

Let E be a vector bundle equipped with a Riemannian metric. We denote by $\Gamma(E)$ the set of all global sections of E. If F is a subbundle of E with the induced metric and $s \in \Gamma(E)$, then $s^F \in \Gamma(F)$ denotes the orthogonal projection of s on F. For a smooth manifold M we denote by TM the tangent bundle of M.

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Now let $\Phi\colon M \looparrowright \bar{M}$ be an immersion of a p-dimensional manifold M in an n-dimensional Riemannian manifold $(\bar{M}, \bar{g})$. Denote by NM the normal bundle to M in $\bar{M}$. Let $\bar{R}(\cdot,\cdot)\cdot$ denote the Riemann curvature tensor on $\bar{M}$. For each $V \in \Gamma(NM)$ define $\mathcal{R}(V) \in \Gamma(NM)$ as the trace of the bilinear form

$$X, Y \mapsto (\bar{R}(X,V)Y)^{NM}.$$

Recall that for each $V \in \Gamma(NM)$ the Weignharten operator

$$A^V\colon \Gamma(TM) \to \Gamma(TM), \quad A^V(X) = -(\nabla_X^{T\bar{M}} V)^{TM}$$

is defined. Let $SM \subset T^*M \otimes TM$ be the bundle of symmetric linear operators on TM. Consider the operator $A\colon \Gamma(NM) \to \Gamma(SM), V \mapsto A^V$. Let $A^*\colon \Gamma(SM) \to \Gamma(NM)$ be the adjoint operator and $\tilde{A} = A^*A$.

Finally, for each $V \in \Gamma(NM)$ consider the operator

$$\nabla^{NM} V\colon \Gamma(TM) \to \Gamma(NM), \quad X \mapsto \nabla_X^{NM} V$$

and the bilinear form

$$X, Y \mapsto -(\nabla_X^{T^*M \otimes NM}(\nabla^{NM} V))(Y) = \nabla^{NM}_{\nabla_X^{TM} Y} V - \nabla_X^{NM}(\nabla_Y^{NM} V)$$

with values in $\Gamma(NM)$. Denote the trace of this bilinear form by $\Delta^{NM} V$ (this is so-called *Laplace-Beltrami operator in normal bundle*). If $e_1, \ldots, e_p$ is a local orthonormal frame in $\Gamma(TM)$, then

$$\Delta^{NM} V = \sum_{i=1}^{p} \nabla^{NM}_{\nabla_{e_i}^{TM} e_i} V - \nabla_{e_i}^{NM}(\nabla_{e_i}^{NM} V). \tag{1}$$

Now suppose that the immersion Φ is minimal and $g = \Phi^*\bar{g}$ is the induced metric on M. Let Φ_t be a variation of Φ and $V := \frac{\partial}{\partial t}\big|_{t=0} \Phi_t \in \Gamma(\Phi^* T\bar{M})$. Then since Φ is minimal, the first variation of the volume functional vanishes: $\frac{d}{dt}\big|_{t=0} \operatorname{Vol}(M, \Phi_t^*\bar{g}) = 0$. The second variation of this functional has the form

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \operatorname{Vol}(M, \Phi_t^*\bar{g}) = \int_M \langle \Delta^{NM}(V^{NM}) + \mathcal{R}(V^{NM}) - \tilde{A}(V^{NM}), V^{NM} \rangle_g \, dv_g,$$

where dv_g stands for the measure associated with the metric g (see [Sim68, Theorems 3.2.1—3.2.2]).

Definition 1. The operator $J\colon \Gamma(NM) \to \Gamma(NM)$, defined by the formula

$$J(V) = \Delta^{NM} V + \mathcal{R}(V) - \tilde{A}(V),$$

is called the *Jacobi (stability) operator* on M.

It is well-known that Jacobi operator is an elliptic differential operator and its spectrum has the form

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots \nearrow +\infty,$$

where each λ_i is listed as many times as its multiplicity. In particular, the number of indices i with $\lambda_i < 0$ is finite.

Definition 2. The number of negative eigenvalues of J counted with multiplicity is called the *index* of M and is denoted by $\operatorname{ind} M$. The multiplicity of the zero eigenvalue of J is called the *nullity* of M and is denoted by $\operatorname{null} M$.

2.2 Bipolar surfaces to Otsuki tori

In this section we shortly describe the construction of bipolar surfaces to Otsuki tori. This construction is based on the Hsiang-Lawson Reduction Theorem [HLJ71, Theorem 2] and is similar to the construction of ordinary Otsuki tori from [Pen13]. Here we give only the most necessary definitions. For more detailed exposition of the construction and its motivation see [Kar14, §2.4]. See also Remark 1.

Let $\mathbb{S}^2$ be the standard unit sphere in $\mathbb{R}^3$ with spherical coordinates (φ, θ) so that

$$\mathbb{S}^2 = \{(\cos\varphi \sin\theta, \cos\varphi\cos\theta, \sin\varphi) \in \mathbb{R}^3 \mid -\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}, 0 \leqslant \theta < 2\pi\},$$

Denote $N=(0,0,1), S=(0,0,-1)\in\mathbb{S}^2$. Consider the metric

$$\tilde g_1 = E\,d\varphi^2 + G\,d\theta^2, \quad E=4\pi^2\cos^2\varphi,\ G=4\pi^2\cos^4\varphi$$

on $\mathbb{S}^2\setminus\{N,S\}$. Let

$$\gamma\colon[0,t_0]\to\mathbb{S}^2, \quad \gamma(t)=(\varphi(t),\theta(t))$$

be a closed geodesic on $(\mathbb{S}^2\setminus\{N,S\},\tilde g_1)$ parametrized by arc length (so that t_0 is the full length of the geodesic). Here we consider $\theta(t)$ modulo 2π.

Proposition 2.2.1. *The surface in $\mathbb{R}^5$ given parametrically by*

$$\begin{cases} x^1(\alpha,t)=\cos\alpha\cos\varphi(t)\sin\theta(t),\\ x^2(\alpha,t)=\sin\alpha\cos\varphi(t)\sin\theta(t),\\ x^3(\alpha,t)=\cos\alpha\cos\varphi(t)\cos\theta(t), \\ x^4(\alpha,t)=\sin\alpha\cos\varphi(t)\cos\theta(t),\\ x^5(\alpha,t)=\sin\varphi(t),\end{cases} \qquad \alpha\in[0,2\pi), t\in[0,t_0), \tag{2}$$

is a minimal torus in $\mathbb{S}^4=\{(x^1,x^2,x^3,x^4,x^5)\in\mathbb{R}^5 \mid \sum_{i=1}^5(x^i)^2=1\}$.

The surface given by (2) is a *bipolar surface to Otsuki tori.*

Now let us describe closed geodesics on $(\mathbb{S}^2\setminus\{N,S\},\tilde g_1)$. To this end, for each $b\in(-\frac{\pi}{2},\frac{\pi}{2})$ consider the geodesic $\gamma_b(t)=(\varphi(t),\theta(t))$ defined by the following initial conditions:

$$\varphi(0)=b,\quad \dot\varphi(0)=0,\quad \theta(0)=0,\quad \dot\theta(0)>0. \tag{3}$$

We want to find out for which values of b this geodesic closes. Note that any closed geodesic on $(\mathbb{S}^2\setminus\{N,S\},\tilde g_1)$ can be parametrized in such a way that conditions (3) hold for some b. Indeed, it suffices to take b equal to the lowest value of the coordinate φ on the geodesic. Thus in such a way we describe all closed geodesics. Moreover, since the metric $\tilde g_1$ is invariant under the transformation $\varphi\mapsto-\varphi$, we may assume that $b\leqslant 0$.

The equation of geodesics for $\ddot\theta$ reads

$$\ddot\theta+\frac{1}{G}\frac{\partial G}{\partial\varphi}\dot\varphi\dot\theta=0 \quad\Leftrightarrow\quad \frac{d}{dt}(G\dot\theta)=0.$$

Hence, $\dot\theta=\frac{c}{G}$ for some constant c. Since the geodesic is naturally parametrized, we get

$$E\dot\varphi^2+G\dot\theta^2=1 \quad\Leftrightarrow\quad \dot\varphi^2=\frac{G-c^2}{EG}.$$

Substituting $t=0$, we obtain $c=\sqrt{G}|_{\varphi=b}=\pm2\pi\cos^2 b$, and since $\dot\theta(0)>0$, we have $c>0$. Hence,

$$\dot\theta=\frac{\cos^2 b}{2\pi\cos^4\varphi} \tag{4}$$

and

$$\dot\varphi^2=\frac{\cos^4\varphi-\cos^4 b}{4\pi^2\cos^6\varphi}. \tag{5}$$

From the last equation it follows that $\dot\varphi$ vanishes exactly when $\varphi=\pm b$ and that the geodesic γ_b lies in the spherical segment $b\leqslant\varphi\leqslant -b$.

Note that there is a special case $b=0$. In this case we have $\varphi(t)\equiv 0, \theta(t)=\frac{t}{2\pi}$ and the surface (2) is just the Clifford torus in equatorial $\mathbb{S}^3\subset\mathbb{S}^5$. Further we assume that $b\neq 0$.

Proposition 2.2.2. *1) The function $\varphi(t)$ is T-antiperiodic where*

$$T=T(b):=\int_b^{-b}\frac{2\pi\cos^3\varphi\,d\varphi}{\sqrt{\cos^4\varphi-\cos^4 b}}<+\infty.$$

2) The function $\varphi(t)$ is odd w.r.t. the transformation $t\mapsto T-t$.
3) The function $\theta(t)$ satisfies $\theta(t+T)\equiv\theta(t)+\theta(T)$.

Proof. The number T is nothing but the difference between the value of t, corresponding to $\varphi = b$, and the nearest value of t, corresponding to $\varphi = -b$. In [Kar14, Corollary 4] it is shown that $T(b) < \sqrt{2}\pi^2 < +\infty$.

A solution of the ODE (5) has two branches. The functions $\varphi(t)$ and $-\varphi(t+T)$ correspond to the same branch of the same solution of (5) (this solution satisfies the initial condition $\varphi(0) = b$). Thus part 1) follows. Part 2) can be proved similarly and part 3) follows easily from part 1) and (4). □

Finally, consider the issue of closeness of the geodesic γ_b. It follows from Proposition 2.2.2 that the geodesic γ_b is closed exactly when the number $\Xi(b) := \theta(T(b))$ is a rational multiple of π. From equations (4),(5) we obtain

$$\Xi(b) = \cos^2 b \int_b^{-b} \frac{d\varphi}{\cos\varphi\sqrt{\cos^4\varphi - \cos^4 b}}.$$

The function $\Xi(b)$ (as well as $T(b)$) is not expressible in terms of elementary functions. However one can prove the following assertions (see [Kar14, Proposition 5]):

(i) $\Xi(b)$ is continuous on $(-\frac{\pi}{2}, 0)$ and strictly increasing;

(ii) $\lim\limits_{b\to 0-} \Xi(b) = \frac{\sqrt{2}}{2}\pi, \quad \lim\limits_{b\to(-\frac{\pi}{2})+} \Xi(b) = \frac{\pi}{2}.$

It follows that closed geodesics on $(\mathbb{S}^2 \setminus \{N,S\}, \tilde{g}_1)$ (and so the bipolar surfaces to Otsuki tori) are in 1-1 correspondence with rational numbers $\frac{p}{q}$ such that

$$p, q > 0, \quad (p,q) = 1, \quad \text{and} \quad \frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2}.$$

We denote the bipolar surface to the Otsuki torus corresponding to the number $\frac{p}{q}$ by $\tilde{O}_{p/q}$. The parameter b of the corresponding closed geodesic γ_b satisfies $\Xi(b) = \frac{p}{q}\pi$ and the full length of this geodesic equals $t_0 = 2qT(b)$.

Remark 1. The above construction *is not* the standard definition of bipolar surfaces to Otsuki tori. According to the standard definition, the surface $\tilde{O}_{p/q}$ is obtained as the *bipolar surface* to some minimal torus in $\mathbb{S}^3$ called *Otsuki torus* and denoted by $O_{p/q}$, see [Kar14, §2.2-2.3]. For details on bipolar surfaces and Otsuki tori see [LJ70, §11] and [Pen13] respectively.

The following proposition contains some important properties of the functions $\varphi(t), \theta(t)$ and immersion (2).

Proposition 2.2.3. *1) The function $\varphi(t)$ has exactly $2q$ zeros on $[0, t_0)$, located at the points $\frac{2d+1}{2}T$ where $d = 0, \dots, 2q-1$. The function $\dot{\varphi}(t)$ has exactly $2q$ zeros on $[0, t_0)$, located at the points dT where $d = 0, \dots, 2q-1$.*

2) The functions $\cos\theta(t)$ and $\sin\theta(t)$ each have exactly $2p$ zeros on $[0, t_0)$.

3) For even q immersion (2) is invariant under the transformation

$$(\alpha, t) \mapsto \left(\alpha + \pi, t + \frac{t_0}{2}\right).$$

The immersion (2) is not invariant under any other transformations.

Proof. Part 1) follows from Proposition 2.2.2 and equation (5). Parts 2) and 3) are proved in [Kar14, Proposition 4]. □

The aim of this paper is to obtain estimates on the index and nullity of the surfaces $\tilde{O}_{p/q}$. A rough upper bound can be obtained as follows. It follows from [EM08, Theorem 1.1] and [Kar21, Proposition 1.6] that

$$\operatorname{ind} \tilde{O}_{p/q} \leqslant 5 \operatorname{ind}_S \tilde{O}_{p/q} + 2,$$

where $\operatorname{ind}_S \tilde{O}_{p/q}$ denotes so-called *spectral index* of $\tilde{O}_{p/q}$, i.e. the number of eigenvalues of the Laplace-Beltrami operator on $\tilde{O}_{p/q}$ less than 2. On the other hand, in [Kar14] it is shown that

$$\operatorname{ind}_S \tilde{O}_{p/q} = \begin{cases} 2q + 4p - 2, & q \text{ is odd;} \\ q + 2p - 2, & q \text{ is even.} \end{cases}$$

It follows immediately that

$$\operatorname{ind} \tilde{O}_{p/q} \leqslant \begin{cases} 10q + 20p - 8, & q \text{ is odd;} \\ 5q + 10p - 8, & q \text{ is even.} \end{cases}$$

Our main result reads as that if $\frac{p}{q}$ is sufficiently close to $\frac{\sqrt{2}}{2}$ (i.e., speaking informally, if $\tilde{O}_{p/q}$ is "close to the Clifford torus"), then one can improve the rough upper bound mentioned above and also obtain a lower bound.

Theorem 1. *There exists $\delta > 0$ such that if $\frac{\sqrt{2}}{2} - \frac{p}{q} < \delta$, then for odd q the inequalities*

$$6q + 8p - 3 \leqslant \operatorname{ind} \tilde{O}_{p/q} \leqslant 10q + 4p - 5$$

hold, and for even q the inequalities

$$3q + 4p - 3 \leqslant \operatorname{ind} \tilde{O}_{p/q} \leqslant 5q + 2p - 5$$

hold. Moreover, for such $\frac{p}{q}$ the inequalities

$$9 \leqslant \operatorname{null} \tilde{O}_{p/q} \leqslant 13$$

hold independently of the parity of q.

2.3 Sturm-Liouville equation

In this section we recall certain properties of Sturm-Liouville equation, which are used in the sequel.

Theorem 2 (Sturm Oscillation Theorem for periodic Sturm-Liouville equation, [CL95, Theorem 3.1 in Chapter 8])**.** *Consider a periodic Sturm-Liouville problem*

$$-(p(t)h'(t))' + q(t)h(t) = \lambda h(t), \tag{6}$$

where

$$p(t) > 0, \quad p(t+T) \equiv p(t), \quad q(t+T) \equiv q(t).$$

Denote by λ_i and $h_i(t)$ (where $i = 0, 1, 2, \ldots$) eigenvalues and eigenfunctions of problem (6) with periodic boundary conditions

$$h(t+T) \equiv h(t).$$

Denote by $\tilde{\lambda}_i$ and $\tilde{h}_i(t)$ (where $i = 1, 2, \ldots$) eigenvalues and eigenfunctions of problem (6) with antiperiodic boundary conditions

$$h(t+T) \equiv -h(t).$$

Then the following inequalities hold:

$$\lambda_0 < \tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 < \lambda_1 \leqslant \lambda_2 < \tilde{\lambda}_3 \leqslant \tilde{\lambda}_4 < \lambda_3 \leqslant \lambda_4 < \ldots.$$

For $\lambda = \lambda_0$ there exists a unique (up to multiplication by a non-zero constant) eigenfunction $h_0(t)$. If $\lambda_{2i+1} < \lambda_{2i+2}$ for some $i \geqslant 0$, then there is a unique (up to multiplication by a non-zero constant) eigenfunction $h_{2i+1}(t)$ with eigenvalue λ_{2i+1} of multiplicity 1 and there is a unique (up to multiplication by a non-zero constant) eigenfunction $h_{2i+2}(t)$ with eigenvalue λ_{2i+2} of multiplicity 1. If $\lambda_{2i+1} = \lambda_{2i+2}$, then there exist two linearly independent eigenfunctions $h_{2i+1}(t)$ and $h_{2i+2}(t)$ with eigenvalue $\lambda = \lambda_{2i+1} = \lambda_{2i+2}$ of multiplicity 2. The same holds in cases $\tilde{\lambda}_{2i+1} < \tilde{\lambda}_{2i+2}$ and $\tilde{\lambda}_{2i+1} = \tilde{\lambda}_{2i+2}$.

The eigenfunction $h_0(t)$ has no zeros on $[0, T)$. The eigenfunctions $h_{2i+1}(t)$ and $h_{2i+2}(t)$ each have exactly $2i + 2$ zeros on $[0, T)$. The eigenfunctions $\tilde{h}_{2i+1}(t)$ and $\tilde{h}_{2i+2}(t)$ each have exactly $2i + 1$ zeros on $[0, T)$.

One more useful fact concerns the case when the coefficients of the equation (6) are periodic with period less than t_0.

Claim 1 ([Kar14, Propositions 11 and 12])**.** *1) Suppose that in the conditions of Theorem 2 the coefficients of the equation (6) are periodic with period $\frac{t_0}{2n}$ for some $n \in \mathbb{Z}, n \geqslant 1$. Then $\frac{t_0}{2n}$-antiperiodic eigenfunctions of the problem (6) are $h_{2n(2k+1)}(t)$ and $h_{2n(2k+1)-1}(t)$, where $k \geqslant 0$.*

2) Suppose that in the conditions of Theorem 2 the coefficients of the equation (6) are periodic with period $\frac{t_0}{n}$ for some $n \in \mathbb{Z}, n \geqslant 2$. Then $\frac{t_0}{n}$-periodic eigenfunctions of the problem (6) are $h_0(t), h_{2nk-1}(t)$ and $h_{2nk}(t)$, where $k \geqslant 1$.

3 Proof of the main theorem

3.1 Jacobi operator on $\tilde{O}_{p/q}$

In this section we compute the Jacobi stability operator J on $\tilde{O}_{p/q}$ in local coordinates α, t.

Proposition 3.1.1. *Let $x \in \tilde{O}_{p/q}$. The following vectors form an orthonormal basis of $T_x\mathbb{R}^5$:*

$$N = \begin{pmatrix} \cos\alpha\cos\varphi\sin\theta \\ \sin\alpha\cos\varphi\sin\theta \\ \cos\alpha\cos\varphi\cos\theta \\ \sin\alpha\cos\varphi\cos\theta \\ \sin\varphi \end{pmatrix},$$

$$e_1 = \frac{1}{\cos\varphi}\frac{\partial}{\partial\alpha} = \begin{pmatrix} -\sin\alpha\sin\theta \\ \cos\alpha\sin\theta \\ -\sin\alpha\cos\theta \\ \cos\alpha\cos\theta \\ 0 \end{pmatrix}, e_2 = 2\pi\cos\varphi\frac{\partial}{\partial t} = 2\pi\cos\varphi\begin{pmatrix} \cos\alpha(-\sin\varphi\sin\theta\,\dot\varphi + \cos\varphi\cos\theta\,\dot\theta) \\ \sin\alpha(-\sin\varphi\sin\theta\,\dot\varphi + \cos\varphi\cos\theta\,\dot\theta) \\ \cos\alpha(-\sin\varphi\cos\theta\,\dot\varphi - \cos\varphi\sin\theta\,\dot\theta) \\ \sin\alpha(-\sin\varphi\cos\theta\,\dot\varphi - \cos\varphi\sin\theta\,\dot\theta) \\ \cos\varphi\,\dot\varphi \end{pmatrix},$$

$$n_1 = \begin{pmatrix} \sin\alpha\cos\theta \\ -\cos\alpha\cos\theta \\ -\sin\alpha\sin\theta \\ \cos\alpha\sin\theta \\ 0 \end{pmatrix}, n_2 = 2\pi\cos\varphi\begin{pmatrix} -\cos\alpha(\cos\theta\,\dot\varphi + \sin\theta\sin\varphi\cos\varphi\,\dot\theta) \\ -\sin\alpha(\cos\theta\,\dot\varphi + \sin\theta\sin\varphi\cos\varphi\,\dot\theta) \\ \cos\alpha(\sin\theta\,\dot\varphi - \cos\theta\sin\varphi\cos\varphi\,\dot\theta) \\ \sin\alpha(\sin\theta\,\dot\varphi - \cos\theta\sin\varphi\cos\varphi\,\dot\theta) \\ \cos^2\varphi\,\dot\theta \end{pmatrix}.$$

Moreover, e_1, e_2 is a basis of $T_x\tilde{O}_{p/q}$ and n_1, n_2 is a basis of $N_x\tilde{O}_{p/q}$.

Proof. The first assertion is easily seen from a direct computation. The vectors e_1 and e_2 lie in $T_x\tilde{O}_{p/q}$ because they are proportional to $\frac{\partial}{\partial\alpha}$ and $\frac{\partial}{\partial t}$ respectively. It remains to note that N is just a unit normal vector to $\mathbb{S}^4$. □

In the sequel, all computations are made w.r.t. the basis from Proposition 3.1.1 and the elements of $\Gamma(N\tilde{O}_{p/q})$ are identified with vector-functions: $f_1n_1 + f_2n_2 \leftrightarrow \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)$.

Proposition 3.1.2. *The matrix of the operator $\tilde{A}\colon \Gamma(N\tilde{O}_{p/q}) \to \Gamma(N\tilde{O}_{p,q})$ in the basis n_1, n_2 is*

$$\tilde{A} = \begin{pmatrix} 8\pi^2\cos^2\varphi\,\dot\theta^2 & 0 \\ 0 & 8\pi^2\sin^2\varphi\cos^2\varphi\,\dot\theta^2 \end{pmatrix}.$$

Proof. We have

$$\langle A^{n_i}e_j, e_k\rangle = -\langle\nabla^{T\tilde{O}_{p/q}}_{e_j}n_i, e_k\rangle = -\langle\nabla^{\mathbb{R}^5}_{e_j}n_i, e_k\rangle = \langle n_i, \nabla^{\mathbb{R}^5}_{e_j}e_k\rangle$$

for all $i, j, k = 1, 2$. Then,

$$\nabla^{\mathbb{R}^5}_{e_1}e_1 = \frac{1}{\cos\varphi}\nabla^{\mathbb{R}^5}_{\frac{\partial}{\partial\alpha}}e_1 = \frac{1}{\cos\varphi}\begin{pmatrix} -\cos\alpha\sin\theta \\ -\sin\alpha\sin\theta \\ -\cos\alpha\cos\theta \\ -\sin\alpha\cos\theta \\ 0 \end{pmatrix}, \nabla^{\mathbb{R}^5}_{e_2}e_1 = 2\pi\cos\varphi\nabla^{\mathbb{R}^5}_{\frac{\partial}{\partial t}}e_1 = 2\pi\cos\varphi\begin{pmatrix} -\sin\alpha\cos\theta\,\dot\theta \\ \cos\alpha\cos\theta\,\dot\theta \\ \sin\alpha\sin\theta\,\dot\theta \\ -\cos\alpha\sin\theta\,\dot\theta \\ 0 \end{pmatrix}. \quad (7)$$

From this it is easy to see that the matrices of the operators A^{n_1}, A^{n_2} in the basis e_1, e_2 are

$$A^{n_1} = \begin{pmatrix} 0 & -2\pi\cos\varphi\,\dot\theta \\ -2\pi\cos\varphi\,\dot\theta & 0 \end{pmatrix}, \quad A^{n_2} = \begin{pmatrix} 2\pi\sin\varphi\cos\varphi\,\dot\theta & 0 \\ 0 & -2\pi\sin\varphi\cos\varphi\,\dot\theta \end{pmatrix}.$$

Here we used that A^{n_1}, A^{n_2} are symmetric and that $\operatorname{tr} A^{n_1} = \operatorname{tr} A^{n_2} = 0$ because the surface $\tilde{O}_{p/q}$ is minimal. Hence, the matrix of the operator A is

$$A = \begin{pmatrix} 0 & 2\pi\sin\varphi\cos\varphi\,\dot\theta \\ -2\pi\cos\varphi\,\dot\theta & 0 \\ -2\pi\cos\varphi\,\dot\theta & 0 \\ 0 & -2\pi\sin\varphi\cos\varphi\,\dot\theta \end{pmatrix},$$

and since $\tilde{A} = A^* A$, the result follows. $\square$

Proposition 3.1.3. *The operator* $\Delta^{N\tilde{O}_{p/q}} \colon \Gamma(N\tilde{O}_{p/q}) \to \Gamma(N\tilde{O}_{p/q})$ *in the basis* n_1, n_2 *has the form*

$$\Delta^{N\tilde{O}_{p/q}} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \Delta f_1 + 4\pi^2 \dot{\varphi}^2 f_1 - \frac{4\pi\dot{\varphi}}{\cos\varphi} \frac{\partial f_2}{\partial \alpha} \\ \Delta f_2 + 4\pi^2 \dot{\varphi}^2 f_2 + \frac{4\pi\dot{\varphi}}{\cos\varphi} \frac{\partial f_1}{\partial \alpha} \end{pmatrix}.$$

Proof. It easily follows from (1) that for any $n \in \Gamma(N\tilde{O}_{p/q})$ and $f \in C^\infty(\tilde{O}_{p/q})$ we have

$$\Delta^{N\tilde{O}_{p/q}}(fn) = f \Delta^{N\tilde{O}_{p/q}} n + (\Delta f) n - 2 \sum_{i=1}^{2} (e_i f) \nabla^{N\tilde{O}_{p/q}}_{e_i} n. \tag{8}$$

Therefore it suffices to calculate $\Delta^{N\tilde{O}_{p/q}} n_i$ and $\nabla^{N\tilde{O}_{p/q}}_{e_j} n_i$ for $i, j = 1, 2$. We have

$$\nabla^{\mathbb{R}^5}_{e_1} n_1 = \frac{1}{\cos\varphi} \nabla^{\mathbb{R}^5}_{\frac{\partial}{\partial\alpha}} n_1 = \begin{pmatrix} \cos\alpha\cos\theta \\ \sin\alpha\cos\theta \\ -\cos\alpha\sin\theta \\ -\sin\alpha\sin\theta \\ 0 \end{pmatrix}, \quad \nabla^{\mathbb{R}^5}_{e_2} n_1 = 2\pi\cos\varphi \nabla^{\mathbb{R}^5}_{\frac{\partial}{\partial t}} n_1 = \begin{pmatrix} -\sin\alpha\sin\theta\,\dot{\theta} \\ \cos\alpha\sin\theta\,\dot{\theta} \\ -\sin\alpha\cos\theta\,\dot{\theta} \\ \cos\alpha\cos\theta\,\dot{\theta} \\ 0 \end{pmatrix}.$$

From this it is easy to see that

$$\langle \nabla^{\mathbb{R}^5}_{e_1} n_2, n_1 \rangle = -\langle \nabla^{\mathbb{R}^5}_{e_1} n_1, n_2 \rangle = 2\pi\dot{\varphi} \quad \text{and} \quad \langle \nabla^{\mathbb{R}^5}_{e_i} n_j, n_k \rangle = 0 \text{ for all other } i, j, k = 1, 2.$$

Hence,

$$\begin{gathered} \nabla^{N\tilde{O}_{p/q}}_{e_1} n_1 = \langle \nabla^{\mathbb{R}^5}_{e_1} n_1, n_2 \rangle n_2 = -2\pi\dot{\varphi}\, n_2, \quad \nabla^{N\tilde{O}_{p/q}}_{e_1} n_2 = \langle \nabla^{\mathbb{R}^5}_{e_1} n_2, n_1 \rangle n_1 = 2\pi\dot{\varphi}\, n_1, \\ \nabla^{N\tilde{O}_{p/q}}_{e_2} n_1 = \nabla^{N\tilde{O}_{p/q}}_{e_2} n_2 = 0. \end{gathered} \tag{9}$$

Further, from (7) we have

$$\nabla^{T\tilde{O}_{p/q}}_{e_1} e_1 = \langle \nabla^{\mathbb{R}^5}_{e_1} e_1, e_2 \rangle e_2 = 2\pi \sin\varphi\, \dot{\varphi}\, e_2, \quad \nabla^{T\tilde{O}_{p/q}}_{e_2} e_2 = -\langle \nabla^{\mathbb{R}^5}_{e_2} e_1, e_2 \rangle e_1 = 0, \tag{10}$$

and using (1), we get

$$\Delta^{N\tilde{O}_{p/q}} n_1 = -\nabla^{N\tilde{O}_{p/q}}_{e_1} \nabla^{N\tilde{O}_{p/q}}_{e_1} n_1 = 4\pi^2 \dot{\varphi}^2 n_1, \quad \Delta^{N\tilde{O}_{p/q}} n_2 = -\nabla^{N\tilde{O}_{p/q}}_{e_1} \nabla^{N\tilde{O}_{p/q}}_{e_1} n_2 = 4\pi^2 \dot{\varphi}^2 n_2, \tag{11}$$

where all other summands from (1) vanish because of (9) and (10). The proposition now follows from (8), (9), (11). $\square$

Proposition 3.1.4 ([Kar14, Proposition 6])**.** *The Laplace-Beltrami operator on* $\tilde{O}_{p/q}$ *is given by the formula*

$$\Delta f = -\frac{1}{\cos^2\varphi} \frac{\partial^2 f}{\partial \alpha^2} - \frac{\partial}{\partial t} \left(4\pi^2 \cos^2\varphi \frac{\partial f}{\partial t} \right).$$

Proposition 3.1.5 ([Sim68, equation (5.1.1)])**.** *The operator* $\mathcal{R} \colon \Gamma(N\tilde{O}_{p/q}) \to \Gamma(N\tilde{O}_{p/q})$ *is the multiplication by* -2*.*

Proposition 3.1.6. *The Jacobi stability operator* $J \colon \Gamma(N\tilde{O}_{p/q}) \to \Gamma(N\tilde{O}_{p/q})$ *in the basis* n_1, n_2 *has the form*

$$J \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \Delta f_1 + 4\pi^2 \dot{\varphi}^2 f_1 - \frac{4\pi\dot{\varphi}}{\cos\varphi} \frac{\partial f_2}{\partial \alpha} - 2 f_1 - 8\pi^2 \cos^2\varphi\, \dot{\theta}^2 f_1 \\ \Delta f_2 + 4\pi^2 \dot{\varphi}^2 f_2 + \frac{4\pi\dot{\varphi}}{\cos\varphi} \frac{\partial f_1}{\partial \alpha} - 2 f_2 - 8\pi^2 \cos^2\varphi \sin^2\varphi\, \dot{\theta}^2 f_1 \end{pmatrix}.$$

Proof. This follows from Propositions 3.1.2–3.1.5. $\square$

3.2 Some elements from the kernel of J

It turns out that some elements of the kernel of J can be found from geometrical considerations. We use the following fact.

Proposition 3.2.1 ([Sim68, Lemmas 5.1.7—5.1.9]). *The image of any Killing vector field on $\mathbb{S}^4$ under the orthogonal projection on $N\tilde{O}_{p/q}$ lies in the kernel of J.*

It is well-known that the space of Killing vector fields on $\mathbb{S}^4$ has dimension 10 and is spanned by the fields

$$x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \quad 1 \leqslant i, j \leqslant 5.$$

The following proposition can be proved by a direct computation.

Proposition 3.2.2. *The image of the space of Killing vector fields on $\mathbb{S}^4$ under the orthogonal projection on $N\tilde{O}_{p/q}$ is spanned by the following vector-functions*

$$\begin{pmatrix}\cos\varphi\sin 2\theta\\ 0\end{pmatrix}, \begin{pmatrix}\cos\varphi\cos 2\theta\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 2\pi\cos^2\varphi\,\dot\varphi\end{pmatrix}, \begin{pmatrix}-\sin\alpha\sin\varphi\cos\theta\\ 2\pi\cos\alpha\cos\varphi(\sin\varphi\cos\theta\,\dot\varphi+\sin\theta\cos\varphi\,\dot\theta)\end{pmatrix},$$
$$\begin{pmatrix}\cos\alpha\sin\varphi\cos\theta\\ 2\pi\sin\alpha\cos\varphi(\sin\varphi\cos\theta\,\dot\varphi+\sin\theta\cos\varphi\,\dot\theta)\end{pmatrix}, \begin{pmatrix}-\sin\alpha\sin\varphi\sin\theta\\ 2\pi\cos\alpha\cos\varphi(\sin\varphi\sin\theta\,\dot\varphi-\cos\theta\cos\varphi\,\dot\theta)\end{pmatrix},$$
$$\begin{pmatrix}\cos\alpha\sin\varphi\sin\theta\\ 2\pi\sin\alpha\cos\varphi(\sin\varphi\sin\theta\,\dot\varphi-\cos\theta\cos\varphi\,\dot\theta)\end{pmatrix}, \begin{pmatrix}-\sin 2\alpha\cos\varphi\\ 2\pi\cos 2\alpha\cos^2\varphi\,\dot\varphi\end{pmatrix}, \begin{pmatrix}\cos 2\alpha\cos\varphi\\ 2\pi\sin 2\alpha\cos^2\varphi\,\dot\varphi\end{pmatrix}.$$

3.3 Separation of variables

Proposition 3.3.1. *Let $h(l,t) = \begin{pmatrix}h_1(l,t)\\ h_2(l,t)\end{pmatrix}$ be a solution of the following periodic matrix Sturm-Liouville problem*

$$\begin{cases} -(4\pi^2\cos^2\varphi\, h_1')' + \left(\dfrac{l^2}{\cos^2\varphi} + 4\pi^2\dot\varphi^2 - 2 - 8\pi^2\cos^2\varphi\,\dot\theta^2\right)h_1 - \dfrac{4\pi l\dot\varphi}{\cos\varphi}h_2 = \lambda h_1,\\ -(4\pi^2\cos^2\varphi\, h_2')' + \left(\dfrac{l^2}{\cos^2\varphi} + 4\pi^2\dot\varphi^2 - 2 - 8\pi^2\sin^2\varphi\cos^2\varphi\,\dot\theta^2\right)h_2 - \dfrac{4\pi l\dot\varphi}{\cos\varphi}h_1 = \lambda h_2,\end{cases} \tag{12}$$

$$h(l,t) \equiv h(l,t+t_0). \tag{13}$$

Then the vector-functions

$$\begin{pmatrix}h_1(l,t)\cos l\alpha\\ h_2(l,t)\sin l\alpha\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix}-h_1(l,t)\sin l\alpha\\ h_2(l,t)\cos l\alpha\end{pmatrix}, \quad l = 0,1,2,\ldots$$

form a basis in the space of eigenfunctions of the operator J with eigenvalue λ. If q is even, then one can chose this basis in such a way that each basis eigenfunction is either $\frac{t_0}{2}$-periodic or $\frac{t_0}{2}$-antiperiodic.

Proof. It suffices to note that the operator J commutes with $\frac{\partial}{\partial\alpha}$ and that for even q the coefficients of the system (12) are $\frac{t_0}{2}$-periodic. □

Denote by $\lambda_k(l)$ the k-th ($k \geqslant 1$) eigenvalue of the problem (12,13). For even q denote by $\lambda_k^+(l)$ (respectively, by $\lambda_k^-(l)$) the k-th ($k \geqslant 1$) eigenvalue of the problem (12,13) for which the corresponding eigenfunction is $\frac{t_0}{2}$-periodic (respectively, $\frac{t_0}{2}$-antiperiodic). Also let us fix the following notation

$$p(t) = 4\pi^2\cos^2\varphi(t),$$

$$Q_l(t) = \begin{pmatrix}\frac{l^2}{\cos^2\varphi} + 4\pi^2\dot\varphi^2 - 2 - 8\pi^2\cos^2\varphi\,\dot\theta^2 & -\frac{4\pi l\dot\varphi}{\cos\varphi}\\ -\frac{4\pi l\dot\varphi}{\cos\varphi} & \frac{l^2}{\cos^2\varphi} + 4\pi^2\dot\varphi^2 - 2 - 8\pi^2\sin^2\varphi\cos^2\varphi\,\dot\theta^2\end{pmatrix}.$$

Proposition 3.3.2. *For $l \geqslant 3$ the inequality $\lambda_1(l) > 0$ holds.*

Proof. Fix arbitrary $l \geqslant 3$. Since

$$\lambda_1(l) = \inf_{\substack{h\in C^\infty(\mathbb{R})\setminus\{0\}\\ h(t)\equiv h(t+t_0)}} \frac{\int_0^{t_0}(p(t)|h'(t)|^2 + \langle h(t), Q_l(t)h(t)\rangle)\,dt}{\int_0^{t_0}|h(t)|^2\,dt} \geqslant \inf_{\substack{h\in C^\infty(\mathbb{R})\setminus\{0\}\\ h(t)\equiv h(t+t_0)}} \frac{\int_0^{t_0}\langle h(t), Q_l(t)h(t)\rangle\,dt}{\int_0^{t_0}|h(t)|^2\,dt},$$

it suffices to show that the matrix $Q_l(t)$ is positive definite for each t. This last assertion holds as soon as the inequality

$$\frac{l^2}{\cos^2\varphi} + 4\pi^2\dot\varphi^2 - 2 - 8\pi^2\cos^2\varphi\,\dot\theta^2 > \left|\frac{4\pi l\dot\varphi}{\cos\varphi}\right|$$

holds for each t. Using (4) and (5), one can rewrite this inequality in the form

$$\left(l - \sqrt{1 - \frac{\cos^4 b}{\cos^4\varphi}}\right)^2 > 2\left(\cos^2\varphi + \frac{\cos^4 b}{\cos^4\varphi}\right).$$

This last inequality holds for $l \geqslant 3$ because in this case we have LHS $\geqslant 4 >$ RHS. □

Thus, the problem (12,13) admits nonpositive eigenvalues only for $l = 0, 1, 2$. In the following sections the cases $l = 0$ and $l = 1, 2$ are considered separately.

3.4 Case $l = 0$

This case is the simplest one because in this case the problem (12,13) is decoupled. We arrive at two scalar periodic Sturm-Liouville problems

$$-(4\pi^2\cos^2\varphi\,h_1')' + (4\pi^2\dot\varphi^2 - 2 - 8\pi^2\cos^2\varphi\,\dot\theta^2)h_1 = \lambda h_1, \tag{14}$$
$$h_1(t + t_0) \equiv h_1(t), \tag{15}$$

and

$$-(4\pi^2\cos^2\varphi\,h_2')' + (4\pi^2\dot\varphi^2 - 2 - 8\pi^2\sin^2\varphi\cos^2\varphi\,\dot\theta^2)h_2 = \lambda h_2, \tag{16}$$
$$h_2(t + t_0) \equiv h_2(t). \tag{17}$$

Our aim is the following

Proposition 3.4.1. *The following inequalities hold*

$$\#\{k \mid \lambda_k(0) < 0\} = 2q + 4p - 1, \quad \#\{k \mid \lambda_k(0) = 0\} = 3.$$

If, in addition, q is even, then

$$\#\{k \mid \lambda_k^+(0) < 0\} = q + 2p - 1, \quad \#\{k \mid \lambda_k^+(0) = 0\} = 3.$$

Denote the k-th ($k \geqslant 0$) eigenvalues of the problems (14,15) and (16,17) by $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ respectively.

Proposition 3.4.2. $\lambda_{4p-1}^{(1)} = \lambda_{4p}^{(1)} = 0$.

Proof. It follows from Proposition 3.2.2 that the functions $\cos\varphi\cos 2\theta$ and $\cos\varphi\sin 2\theta$ are the eigenfunctions of the problem (14,15) with eigenvalue 0. These functions are linearly independent and each have exactly $4p$ zeros on $[0, t_0)$ (see Proposition 2.2.3), hence the proposition follows from Theorem 2. □

Now consider the problem (16,17). Unfortunately, for this problem Proposition 3.2.2 gives only one eigenfunction $2\pi\cos^2\varphi\,\dot\varphi$ and we cannot determine the number of the eigenvalue 0 using only Theorem 2. However, this eigenfunction is T-antiperiodic and the coefficients of the equation (16) are periodic with the same period. Therefore it is natural to consider the following antiperiodic Sturm-Liouville problem

$$-(4\pi^2\cos^2\varphi\,h_2')' + (4\pi^2\dot\varphi^2 - 2 - 8\pi^2\sin^2\varphi\cos^2\varphi\,\dot\theta^2)h_2 = \lambda h_2, \tag{18}$$
$$h_2(t + T) \equiv -h_2(t). \tag{19}$$

Denote the k-th ($k \geqslant 1$) eigenvalue of the problem (18,19) by $\tilde\lambda_k^{(2)}$.

Proposition 3.4.3. $\tilde{\lambda}_1^{(2)} < 0, \tilde{\lambda}_2^{(2)} = 0$.

Proof. It follows from Proposition 3.2.2 that the function $2\pi \cos^2 \varphi \, \dot{\varphi}$ is the eigenfunction of the problem (18,19) with eigenvalue 0. Since on the segment $[0, T]$ this function vanishes only at the points $t = 0, T$, it follows from Theorem 2 that either $\tilde{\lambda}_1^{(2)} = 0$ or $\tilde{\lambda}_2^{(2)} = 0$.

Let us show that in fact $\tilde{\lambda}_2^{(2)} = 0$. One can verify by a direct computation that the substitution $z(t) = \sqrt{p(t)} h_2(t) = 2\pi \cos \varphi(t) \, h_2(t)$ transforms the problem (16,17) into the problem

$$-z'' + \frac{\lambda + 2}{p(t)} z = 0, \tag{20}$$

$$z(t + T) \equiv -z(t). \tag{21}$$

We need the following claim, which one can consider as the integral version of the permutation inequality.

Claim 2. *Let $f(t)$ and $g(t)$ be two (non-strictly) monotonic functions on $[0, a]$ with different kinds of monotonicity. Then*

$$\int_0^a f(t) g(t) \, dt \leqslant \int_0^a f(t) g(a - t) \, dt,$$

where the equality is attained exactly when at least one of the functions f, g is a constant function.

Proof.

$$\int_0^a f(t)(g(t) - g(a - t)) \, dt = \int_0^{\frac{a}{2}} f(t)(g(t) - g(a - t)) \, dt + \int_{\frac{a}{2}}^a f(t)(g(t) - g(a - t)) \, dt =$$

$$\int_0^{\frac{a}{2}} f(t)(g(t) - g(a - t)) \, dt - \int_0^{\frac{T}{2}} f(a - t)(g(a - t) - g(t)) \, dt = \int_0^{\frac{a}{2}} (f(t) - f(a - t))(g(t) - g(a - t)) \, dt \leqslant 0,$$

where the last inequality follows from the monotonicity of f and g. The equality happens exactly when the integrand is identically zero. In such a case we have $f(0) = f(a)$ or $g(0) = g(a)$ and from the monotonicity we have $f = \text{const}$ or $g = \text{const}$. □

Let us continue the proof of Proposition 3.4.3. The function $\zeta(t) = 2\pi \cos^3 \varphi(t) \, \dot{\varphi}(t)$ solves the problem (20,21) with $\lambda = 0$. Note that the functions $p(t)$ and $\zeta(t)$ are strictly increasing on $[0, \frac{T}{2}]$. Using $\zeta(\frac{T}{2} - t)$ as a test function, we find

$$\tilde{\lambda}_1^{(2)} + 2 \leqslant \frac{\int_0^T \zeta'(\frac{T}{2} - t)^2 \, dt}{\int_0^T p(t)^{-1} \zeta(\frac{T}{2} - t)^2 \, dt} = \frac{\int_0^{\frac{T}{2}} \zeta'(t)^2 \, dt}{\int_0^{\frac{T}{2}} p(t)^{-1} \zeta(\frac{T}{2} - t)^2 \, dt} < \frac{\int_0^{\frac{T}{2}} \zeta'(t)^2 \, dt}{\int_0^{\frac{T}{2}} p(t)^{-1} \zeta(t)^2 \, dt} = \frac{\int_0^T \zeta'(t)^2 \, dt}{\int_0^T p(t)^{-1} \zeta(t)^2 \, dt} = 2.$$

Here we used Claim 2 and the fact that ζ is even w.r.t. the transformation $t \mapsto T - t$. □

Proposition 3.4.4. $\lambda_{2q-1}^{(2)} = \tilde{\lambda}_1^{(2)} < 0, \lambda_{2q}^{(2)} = \tilde{\lambda}_2^{(2)} = 0$.

Proof. This follows from Proposition 3.4.3 and assertion 1) of Proposition 1 for $n = q$. □

Proof of Proposition 3.4.1. This follows from Propositions 3.4.2, 3.4.4 and Claim 1 for $n = 2$. □

Remark 2. The trick, used in the proof of Proposition 3.4.3, can be used to simplify the proof of [Kar14, §3.4].

3.5 Preparation for the cases $l = 1$ and $l = 2$

In this section we describe our approach for more complicated cases $l = 1$ and $l = 2$, in which the system (12) is not decoupled.

Consider the following matrix Sturm-Liouville problem

$$\begin{cases} -(4\pi^2 \cos^2 \varphi \, h_1')' + \left(\dfrac{l^2}{\cos^2 \varphi} + 4\pi^2 \dot{\varphi}^2 - 2 - 8\pi^2 \cos^2 \varphi \, \dot{\theta}^2 \right) h_1 - \dfrac{4\pi l \dot{\varphi}}{\cos \varphi} h_2 = \lambda h_1, \\ -(4\pi^2 \cos^2 \varphi \, h_2')' + \left(\dfrac{l^2}{\cos^2 \varphi} + 4\pi^2 \dot{\varphi}^2 - 2 - 8\pi^2 \sin^2 \varphi \cos^2 \varphi \, \dot{\theta}^2 \right) h_2 - \dfrac{4\pi l \dot{\varphi}}{\cos \varphi} h_1 = \lambda h_2, \end{cases} \tag{22}$$

$$h_1(t + T) \equiv \omega h_1(t), \quad h_2(t + T) \equiv -\omega h_2(t), \tag{23}$$

where ω is some $2q$-th root of unity and $h(t)$ is a complex-valued vector-function. Denote the k-th ($k \geqslant 1$) eigenvalue of the problem (22,23) by $\lambda_k^{[\omega]}(l)$.

Proposition 3.5.1. *Denote $\varepsilon = e^{\frac{1}{q}\pi i}$. Then*

$$\{\lambda_k(l) \mid k \geqslant 1\} = \bigcup_{r=0}^{2q-1} \{\lambda_k^{[\varepsilon^r]}(l) \mid k \geqslant 1\},$$

and if q is even, then

$$\{\lambda_k^+(l) \mid k \geqslant 1\} = \bigcup_{r=0}^{q-1} \{\lambda_k^{[\varepsilon^{2r}]}(l) \mid k \geqslant 1\}, \quad \{\lambda_k^-(l) \mid k \geqslant 1\} = \bigcup_{r=0}^{q-1} \{\lambda_k^{[\varepsilon^{2r+1}]}(l) \mid k \geqslant 1\}.$$

Here we mean the equalities of multisets, i.e. each eigenvalue is counted with multiplicity.

Proof. This is standard. It suffices to note that the cyclic group of order $2q$ acts on the solution space of the problem (12,13) by the rule

$$\sigma * \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \varepsilon \begin{pmatrix} h_1(t) \\ -h_2(t) \end{pmatrix}, \tag{24}$$

where σ is a generator of the cyclic group. Thus the first assertion of the proposition follows easily from the existence of a joint basis of eigenfunctions. For the second assertion just note that an eigenfunction of (22,23) is $\frac{t_0}{2}$-antiperiodic exactly when $\omega^q = 1$. $\square$

Proposition 3.5.1 allows us to reduce the counting of negative eigenvalues of the problem (12,13) to the counting of negative eigenvalues of the problems (22,23) where ω runs through all $2q$-th roots of unity. For the problems (22,23) we use an approach based on the work of Edwards [Edw64].

Denote by $\mu_k(l)$ the k-th eigenvalue of the Dirichlet problem

$$\begin{cases} -(4\pi^2\cos^2\varphi\, h_1')' + \left(\dfrac{l^2}{\cos^2\varphi} + 4\pi^2\dot\varphi^2 - 2 - 8\pi^2\cos^2\varphi\,\dot\theta^2\right) h_1 - \dfrac{4\pi l\dot\varphi}{\cos\varphi} h_2 = \mu h_1, \\ -(4\pi^2\cos^2\varphi\, h_2')' + \left(\dfrac{l^2}{\cos^2\varphi} + 4\pi^2\dot\varphi^2 - 2 - 8\pi^2\sin^2\varphi\cos^2\varphi\,\dot\theta^2\right) h_2 - \dfrac{4\pi l\dot\varphi}{\cos\varphi} h_1 = \mu h_2, \\ h(0) = h(T) = 0. \end{cases}$$

Also denote

$$\Omega_l[g,h] = \int_0^T (p(t)\langle g'(t), h'(t)\rangle + \langle g(t), Q_l(t)h(t)\rangle)dt =$$
$$\langle g(t), p(t)h'(t)\rangle\Big|_0^T + \int_0^T \langle g(t), -(p(t)h'(t))' + Q_l(t)h(t)\rangle dt,$$

where $\langle\cdot,\cdot\rangle$ stands for the standard Hermitian product in $\mathbb{C}^2$.

Let S_l be the solution space of the system (22) on the segment $[0,T]$. Consider the map

$$S_l \to \mathbb{C}^2 \oplus \mathbb{C}^2, \quad h(t) \mapsto (h(0), h(T)). \tag{25}$$

Suppose that $\mu_k(l) \neq 0$ for each $k \geqslant 1$ (we will see that this is the case at least for b close to zero). Then the kernel of the map (25) is trivial and since $\dim S_l = 4$, we see that this map is a bijection. Define the Hermitian form $\alpha_l[\cdot]$ on $\mathbb{C}^2 \oplus \mathbb{C}^2$ by the formula

$$\alpha[(v_0, v_T)] = \Omega_l[h,h],$$

where h is a unique element of S_l such that $h(0) = v_0, h(T) = v_T$. We denote the corresponding sesquilinear form by $\alpha_l[\cdot,\cdot]$. Finally, let us fix the following basis of $\mathbb{C}^2 \oplus \mathbb{C}^2$

$$e_1 = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right), \quad e_2 = \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right), \quad e_3 = \left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right), \quad e_4 = \left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right).$$

Proposition 3.5.2 (see [Edw64, Proposition 2.6]). *Suppose that $\mu_k(l) \neq 0$ for each $k \geqslant 1$. Then*

$$\#\{k \mid \lambda_k^{[\omega]}(l) < 0\} = \#\{k \mid \mu_k(l) < 0\} + \operatorname{ind}(\alpha_l|\Pi_\omega),$$
$$\#\{k \mid \lambda_k^{[\omega]}(l) = 0\} = \operatorname{null}(\alpha_l|\Pi_\omega),$$

where $\Pi_\omega = \operatorname{span}\{e_1 + \omega e_3, e_2 - \omega e_4\}$, vertical bar denotes restriction, and ind, null *denote the index and the nullity of a Hermitian form on a subspace respectively.*

Denote $a_{ij} = \alpha_l[e_i, e_j]$. Since the coefficients of the system (22) are real-valued functions, we have $a_{ij} \in \mathbb{R}$, in particular, $a_{ij} = a_{ji}$. Moreover, since the coefficients of the system (22) are even functions w.r.t. the transformation $t \mapsto T - t$, we obtain

$$a_{ij} = a_{\sigma(i)\sigma(j)}, \text{ where } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

Therefore the Gram matrix of the restriction $\alpha_l|\Pi_\omega$ has the form

$$A_l(\omega) = \begin{pmatrix} a_{11} + (\omega + \bar\omega)a_{13} + a_{33} & a_{12} + \omega a_{23} - \bar\omega a_{14} - a_{34} \\ a_{12} + \bar\omega a_{23} - \omega a_{14} - a_{34} & a_{22} - (\omega + \bar\omega)a_{24} + a_{44} \end{pmatrix} =$$
$$= \begin{pmatrix} 2a_{11} + (\omega + \bar\omega)a_{13} & (\omega - \bar\omega)a_{14} \\ (\bar\omega - \omega)a_{14} & 2a_{22} - (\omega + \bar\omega)a_{24} \end{pmatrix}. \tag{26}$$

In particular, $\det A_l(\omega)$ is a polynomial in $s = \operatorname{Re}\omega$ of degree at most 2. We denote this polynomial by $P_l(s)$.

Now consider the case of Clifford torus $b = 0$. In this case we have

$$\varphi(t) \equiv 0, \quad \theta(t) = \frac{t}{2\pi}, \quad T = \lim_{b\to 0} T(b) = \sqrt{2}\pi^2.$$

The problem (22,23) becomes

$$\begin{cases} -4\pi^2 h_1'' + (l^2 - 4)h_1 = \lambda h_1, \\ -4\pi^2 h_2'' + (l^2 - 2)h_2 = \lambda h_2, \end{cases} \tag{27}$$
$$h_1(t + T) \equiv \omega h_1(t), \quad h_2(t + T) \equiv -\omega h_2(t). \tag{28}$$

Since the problem (27,28) is decoupled, it is easy to see that

$$\mu_1(1) = -1, \ \mu_2(1) = 1 > 0, \quad \mu_1(2) = 2 > 0. \tag{29}$$

In particular, $\mu_k(l) \neq 0$ for each $k \geqslant 0$. Therefore the matrices $A_l(\omega), l = 1, 2$ are defined and can be computed.

Proposition 3.5.3. *For $b = 0$ we have*

$$A_1(\omega) = \begin{pmatrix} \frac{4\sqrt{3}\pi}{\sin\frac{\sqrt{6}}{2}\pi}(\cos\frac{\sqrt{6}}{2}\pi - \operatorname{Re}\omega) & 0 \\ 0 & \frac{4\pi}{\sin\frac{\sqrt{2}}{2}\pi}(\cos\frac{\sqrt{2}}{2}\pi + \operatorname{Re}\omega) \end{pmatrix},$$
$$A_2(\omega) = \begin{pmatrix} 4\sqrt{2}(1 - \operatorname{Re}\omega) & 0 \\ 0 & \frac{4\sqrt{2}\pi}{\sinh\pi}(\cosh\pi + \operatorname{Re}\omega) \end{pmatrix}.$$

Proof. We consider only the case $l = 1$ because the case $l = 2$ is similar. Let $\psi_i(t)$ be the solution of the system (27) satisfying $(\psi_i(0), \psi_i(T)) = e_i$ $(i = 1, 2, 3, 4)$. On can verify by a direct computation that

$$\psi_1 = \frac{1}{\sin\frac{\sqrt{6}}{2}\pi}\begin{pmatrix} \sin\frac{\sqrt{3}(T-t)}{2\pi} \\ 0 \end{pmatrix}, \qquad \psi_2 = \frac{1}{\sin\frac{\sqrt{2}}{2}\pi}\begin{pmatrix} 0 \\ \sin\frac{T-t}{2\pi} \end{pmatrix},$$
$$\psi_3 = \frac{1}{\sin\frac{\sqrt{6}}{2}\pi}\begin{pmatrix} \sin\frac{\sqrt{3}t}{2\pi} \\ 0 \end{pmatrix}, \qquad \psi_4 = \frac{1}{\sin\frac{\sqrt{2}}{2}\pi}\begin{pmatrix} 0 \\ \sin\frac{t}{2\pi} \end{pmatrix}.$$

Since

$$a_{ij} = \alpha_1[e_i, e_j] = \Omega_1[\psi_i, \psi_j] = \langle \psi_i, p\psi_j' \rangle\Big|_0^T,$$

we see that all the values a_{ij} can be easily computed. ☐

Hence for $b=0$ we have

$$\operatorname{ind}(\alpha_1|\Pi_\omega)=\begin{cases}2, & \operatorname{Re}\omega\in[-1,\cos\frac{\sqrt{6}}{2}\pi),\\ 1, & \operatorname{Re}\omega\in[\cos\frac{\sqrt{6}}{2}\pi,-\cos\frac{\sqrt{2}}{2}\pi),\\ 0, & \operatorname{Re}\omega\in[-\cos\frac{\sqrt{2}}{2}\pi,1],\end{cases}\qquad \operatorname{null}(\alpha_1|\Pi_\omega)=\begin{cases}1, & \operatorname{Re}\omega=\cos\frac{\sqrt{6}}{2}\pi,-\cos\frac{\sqrt{2}}{2}\pi,\\ 0, & \text{otherwise.}\end{cases}$$

$$\operatorname{ind}(\alpha_2|\Pi_\omega)=0,\qquad \operatorname{null}(\alpha_2|\Pi_\omega)=\begin{cases}1, & \omega=1,\\ 0, & \text{otherwise.}\end{cases}$$

We need the following technical proposition.

Proposition 3.5.4. *The values a_{ij} and $\mu_k(l)$ depend continuously on b.*

Proof. The substitution $t\mapsto\frac{t}{T}$ allows us to consider the system (22) as a system on the segment $[-1,1]$ independent of b. Then the continuity of a_{ij} as a function of b follows easily from the continuous dependence of the solution of a system of ODEs on the coefficients. The continuity of the Dirichlet eigenvalues $\mu_k(l)$ follows, for instance, from [HLWZ19, Theorem 6]. □

3.6 Estimates for the cases $l=1$ and $l=2$

Proposition 3.6.1. *For each b sufficiently close to zero we have*

$$\begin{aligned}2q+2p-1\leqslant&\#\{k\mid\lambda_k(1)<0\}\leqslant 4q-2,\\ 2\leqslant&\#\{k\mid\lambda_k(1)=0\}\leqslant 4.\end{aligned}\tag{30}$$

If, in addition, q is even, then

$$\begin{aligned}q+p-1\leqslant&\#\{k\mid\lambda_k^-(1)<0\}\leqslant 2q-2,\\ 2\leqslant&\#\{k\mid\lambda_k^-(1)=0\}\leqslant 4.\end{aligned}\tag{31}$$

Proof. First note that it follows from (29) and Proposition 3.5.4 that for b sufficiently close to zero we have

$$\#\{k\mid\mu_k(1)<0\}=1.\tag{32}$$

When $b=0$ the roots of the polynomial $P_1(s)$ are $\cos\frac{\sqrt{6}}{2}\pi$ and $-\cos\frac{\sqrt{2}}{2}\pi$. It follows from Proposition 3.5.4 that for b sufficiently close to zero the roots s_1 and s_2 of the polynomial $P_1(s)$ are close to $\cos\frac{\sqrt{6}}{2}\pi$ and $-\cos\frac{\sqrt{2}}{2}\pi$ respectively and we have

$$\operatorname{ind}(\alpha_1|\Pi_\omega)=\begin{cases}2, & \operatorname{Re}\omega\in[-1,s_1),\\ 1, & \operatorname{Re}\omega\in[s_1,s_2),\\ 0, & \operatorname{Re}\omega\in[s_2,1],\end{cases}\qquad \operatorname{null}(\alpha_1|\Pi_\omega)=\begin{cases}1, & \operatorname{Re}\omega=s_1,s_2,\\ 0, & \text{otherwise.}\end{cases}\tag{33}$$

At the same time, it follows from Proposition 3.2.2 that the vector-function

$$e^{i\theta}\begin{pmatrix}\sin\varphi\\ 2\pi\cos\varphi(\sin\varphi\,\dot\varphi-i\cos\varphi\,\dot\theta)\end{pmatrix}=\begin{pmatrix}\sin\varphi\cos\theta\\ 2\pi\cos\varphi(\sin\varphi\cos\theta\,\dot\varphi+\sin\theta\cos\varphi\,\dot\theta)\end{pmatrix}+\\ +i\begin{pmatrix}\sin\varphi\sin\theta\\ 2\pi\cos\varphi(\sin\varphi\sin\theta\,\dot\varphi-\cos\theta\cos\varphi\,\dot\theta)\end{pmatrix}$$

solves the problem (22,23) with $l=1,\lambda=0,\omega=-e^{\frac{p}{q}\pi i}$. Therefore it follows from Proposition 3.5.2 that $\det A_1(-e^{\frac{p}{q}\pi i})=0$, i.e.

$$s_2=-\cos\frac{p}{q}\pi.\tag{34}$$

Note that for b sufficiently close to zero we have $|s_1|>s_2$. Thus from (33), (34) we have

$$2p-1\leqslant\sum_{r=0}^{2q}\operatorname{ind}(\alpha_1|\Pi_{\varepsilon^r})\leqslant 2q-2,\quad 2\leqslant\sum_{r=0}^{2q}\operatorname{null}(\alpha_1|\Pi_{\varepsilon^r})\leqslant 4,\tag{35}$$

and for even q we have

$$p-1 \leqslant \sum_{r=0}^{q-1} \operatorname{ind}(\alpha_1|\Pi_{\varepsilon^{2r+1}}) \leqslant q-2, \quad 2 \leqslant \sum_{r=0}^{q-1} \operatorname{null}(\alpha_1|\Pi_{\varepsilon^{2r+1}}) \leqslant 4. \tag{36}$$

The required inequalities (30) and (31) follow now from (32), (35), (36) and Propositions 3.5.1, 3.5.2. □

Proposition 3.6.2. *For any b sufficiently close to zero we have*

$$\lambda_1(2)=0, \quad \lambda_2(2)>0.$$

If, in addition, q is even, then

$$\lambda_1^+(2)=0, \quad \lambda_2^+(2)>0.$$

Proof. This is similar to the proof of Proposition 3.6.1 but easier. Again we note that it follows from (29) and Proposition 3.5.4 that for b sufficiently close to zero we have

$$\#\{k \mid \mu_k(2)<0\}=0. \tag{37}$$

For $b=0$ the roots of the polynomial $P_2(s)$ are 1 and $-\cosh\pi$. At the same time, it follows from Proposition 3.2.2 that the vector-function $\left(\begin{smallmatrix}\cos\varphi \\ 2\pi\cos^2\varphi\,\dot\varphi\end{smallmatrix}\right)$ solves the problem (22,23) with $l=2, \lambda=0, \omega=1$. Therefore for each b the polynomial $P_2(s)$ has the root 1 and for b sufficiently close to zero the second root of $P_2(s)$ is close to $-\cosh\pi<-1$. Thus for b sufficiently close to zero we have

$$\operatorname{ind}(\alpha_2|\Pi_\omega)=0, \quad \operatorname{null}(\alpha_2|\Pi_\omega)=\begin{cases}1, & \omega=1,\\ 0, & \text{otherwise.}\end{cases} \tag{38}$$

The proposition now follows from (37), (38) and Propositions 3.5.1, 3.5.2. □

3.7 Estimates on index and nullity

Proof of Theorem 1. Let us put together the results of §3.4 and §3.6. Suppose that q is odd. According to Proposition 3.3.1 each eigenfunction of the problem (12,13) with $l=0$ and $\lambda<0$ ($\lambda=0$) contributes 1 to $\operatorname{ind}\tilde O_{p/q}$ ($\operatorname{null}\tilde O_{p/q}$), and each eigenfunction of the problem (12,13) with $l=1,2$ and $\lambda<0$ ($\lambda=0$) contributes 2 to $\operatorname{ind}\tilde O_{p/q}$ ($\operatorname{null}\tilde O_{p/q}$). Hence,

$$\operatorname{ind}\tilde O_{p/q}=\#\{k\mid\lambda_k(0)<0\}+2\#\{k\mid\lambda_k(1)<0\}+2\#\{k\mid\lambda_k(2)<0\},$$
$$\operatorname{null}\tilde O_{p/q}=\#\{k\mid\lambda_k(0)=0\}+2\#\{k\mid\lambda_k(1)=0\}+2\#\{k\mid\lambda_k(2)=0\},$$

and from Propositions 3.4.1, 3.6.1, 3.6.2 we have

$$6q+8p-3=(2q+4p-1)+2(2q+2p-1)\leqslant\operatorname{ind}\tilde O_{p/q}\leqslant(2q+4p-1)+2(4q-2)=10q+4p-5,$$
$$9=3+2\cdot2+2\cdot1\leqslant\operatorname{null}\tilde O_{p/q}\leqslant 3+2\cdot4+2\cdot1=13.$$

Suppose that q is even. Then the immersion (2) is invariant under the transformation

$$(\alpha,t)\mapsto\left(\alpha+\pi,t+\frac{t_0}{2}\right).$$

It follows that for even (respectively, odd) l only $\frac{t_0}{2}$-periodic (respectively, $\frac{t_0}{2}$-antiperiodic) eigenfunctions of the problem (12,13) contribute to index and nullity. Hence,

$$\operatorname{ind}\tilde O_{p/q}=\#\{k\mid\lambda_k^+(0)<0\}+2\#\{k\mid\lambda_k^-(1)<0\}+2\#\{k\mid\lambda_k^+(2)<0\},$$
$$\operatorname{null}\tilde O_{p/q}=\#\{k\mid\lambda_k^+(0)=0\}+2\#\{k\mid\lambda_k^-(1)=0\}+2\#\{k\mid\lambda_k^+(2)=0\},$$

and from Propositions 3.4.1, 3.6.1, 3.6.2 we have

$$3q+4p-3=(q+2p-1)+2(q+p-1)\leqslant\operatorname{ind}\tilde O_{p/q}\leqslant(q+2p-1)+2(2q-2)=5q+2p-5,$$
$$9=3+2\cdot2+2\cdot1\leqslant\operatorname{null}\tilde O_{p/q}\leqslant 3+2\cdot4+2\cdot1=13.$$

□

Remark 3. 1) Unfortunately, our method does not allow us to obtain explicit estimates on δ.

2) We suppose that finding the *exact* value of $\operatorname{ind}\tilde{O}_{p/q}$ for each $\frac{p}{q}$ is a very difficult problem. Indeed, such a computation at least should contain a very accurate control over the root s_1 from the proof of Proposition 3.6.1, while this root hardly can be calculated exactly.

3) A numerical experiment in Wolfram Mathematica allows us to conjecture that

$$\operatorname{ind}\tilde{O}_{2/3} = 31, \quad \operatorname{null}\tilde{O}_{2/3} = 9,$$

which coincides with the lower bounds from Theorem 1. However it is easy to see from the proof of Theorem 1 that for some of the surfaces $\tilde{O}_{p/q}$ this is not the case. Indeed, analyzing the proof of Proposition 3.6.1, one can see that the lower bound on $\operatorname{ind}\tilde{O}_{p/q}$ can be attained only if $s_1 < -1$ while for b sufficiently close to zero the inequality $s_1 > -1$ holds.

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